Social Security and Macroeconomic Risk in General Equilibrium

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Contents

Abstract

This paper studies the interaction between macro-economic risks, and paygo social security. For this, it uses an applied general equilibrium model with overlapping generations of risk-averse households. The sources of risk are productivity shocks and capital return shocks. The risk profile of pensions differs from that of financial assets, because pensions are linked partially to future wage rates and productivity. The model is used to discuss the effects social security on labor supply, private saving, and welfare in a closed economy. Results show that the welfare effects of paygo social security are negative as crowding out dominates the positive insurance effects.

1 Introduction

In the assessment of pension systems, it is important to distinguish the financial sustainability aspect from the risk-sharing aspects of pension systems. The rise in old-age dependency ratios over the next couple of decades will substantially shrink the contribution base of pension funds relative to the base of recipients. This implies ever increasing contribution rates, that must at some point be quenched by reforms to the existing scheme. However, the lack of sustainability of a pension scheme does not in itself imply a risk. A risk arises only if the timing or direction of the reform is uncertain. For example, the U.S. social security fund is expected to run out of funds by 2040. Several European pension systems are considered to be unsustainable as well. A postponement of policy adjustments for these schemes creates substantial uncertainty among participants in the schemes both with respect to future contribution rates *and* with respect to the real value of future pension benefits.

The adjustments that have been made so far show a general movement towards a Defined Contribution (DC) system, in which the contribution rate is fixed, and benefits are uncertain. This change contrasts strongly with the original purpose of collective pension systems, where the benefit was defined (a DB system) so as to guarantee retired workers a minimum level of welfare. In addition, a shift can be observed from collective schemes towards private saving accounts, which reduces the role of collective risk sharing in exchange for a larger element of private risk. Future pensions are increasingly at risk and the general public is becoming aware of this.

In this study, I address the question how a *sustainable* PAYG pension scheme distributes risk among generations and what value these generations attach to this risk sharing. This question is made more relevant by a demographic shift, as it increases pension risk through smaller contribution bases. However, the demographic shift itself does not necessarily constitute a risk factor for pension provisions. Rather, rising dependency ratios make the social security system more vulnerable to macroeconomic shocks in general. These shocks may be demographic in character, but they can also originate with asset market returns or productivity risk. To the extent that the rise in dependency ratios is predictable, these risk factors are more important for the viability of pension systems than demographic shocks.

Only a few studies address the macroeconomic risk sharing aspects of social security in a general equilibrium framework. Brooks (2000) analyses the role of a Defined Contribution PAYG social security system. He concludes that this type of social security system does not provide much insurance, because PAYG benefits are positively correlated with asset market returns.[1] Krueger and Kubler (2006) analyse the efficiency effects of a Defined Contribution un-

[1]This is in line with the theoretical study of Bohn (1999b) , who concludes that a pure DC system offers too little insurance to the old, while a pure DB system offers too much insurance.

funded social security system in an economy with both productivity risk and capital return risk. Sánchez-Marcos and Sánchez-Martín (2006) analyse an economy with population growth risk (fertility risk) and a Defined Benefit unfunded social security system. Both studies conclude that the gains from intergenerational risk sharing do not compensate for the adverse crowding out effects. Part of the adverse effects of social security occurs through the general equilibrium effects on factor prices. However, Miles and Cerny (2006) study the optimal PAYG component of social security for a small open economy (Japan) with exogenous labour supply. The trade-off is in terms of the balance between funded defined-contribution private saving accounts and unfunded defined-benefit state pensions. The main conclusion of their study too is that in the long-run the adverse effects of crowding out of private saving dominate the efficiency gain of the additional insurance of a state pension, so that virtually everybody is better off with private saving accounts. These conclusions are at variance with those of Matsen and Thøgersen (2004), possibly because the latter use a partial equilibrium framework that does not consider crowding out issues.

This paper focuses on the interaction between macro-economic risks and social security in an ageing society. To this end, the paper employs an applied general equilibrium model to describe macro-economic risk and the response of economic agents to risk. Important sources of risk are productivity shocks and interest rate shocks. In the absence of a complete system of asset markets, households will value social security if it provides them with a quasi-asset that allows them to better diversify their old-age income risk. In the absence of a market for wage-indexed bonds, such an asset may be provided by a wage-indexed paygo scheme. A Defined Benefit paygo scheme that links benefits to wages offers a form of productivity risk sharing between old and young generations.

The paper uses a stochastic CGE model to address these issues. The stochastic properties of the model derive from uncertainty about the rate of depreciation of capital and labour productivity. The return to capital depends both on depreciation shocks and labour productivity. In addition to capital, households can also trade claims on a one-period risk-free bond. In addition, households have an implicit claim on social security, which functions like a non-tradable asset in the decisions of households. Households have separate consumption smoothing incentives and risk diversification motives, which are modelled through a non-expected utility function.

The calibration delivers a setting with fairly impatient households, who initially do not want to save in either bonds or equity. The lack of a positive equity portfolio is due mostly to the substantial correlation between long-term returns to equity and bonds. Given that young households face a rising wage profile, they shift forward their future labour income and initially run a financial debt. However, short selling of equity is impossible, as returns to capital are unbounded. A negative equity portfolio thus creates a risk of insolvency, which is not allowed in this model. So yound households only hold a negative position in bonds, and have zero equity. As a result, the model shows an equity premium of approximately 3%, given an Arrow-Pratt relative risk aversion of 3.

The government levies distortionary taxes that are redistributed in a lump-sum fashion to households. The size of the lump-sum payments is indexed to wages. Government fiscal policy is a simple balanced-budget rule, which implies that tax rates fluctuate randomly in response to fluctuations in tax receipts. Social security is initially modelled as a DB paygo system that offers a fixed replacement rate to pensioners in terms of the after-tax real wage. Two policy options are investigated, a shift from DB to DC, and a trimming down of the PAYG pension, with compensation for current pension rights.

The model used in this paper resembles that of Krueger and Kubler (2006). The main differences are that labour supply is endogenous in the present model, that shocks are lognormally distributed, so that shocks are not bounded, and that the OLG model is an *annual* one, in which households are distinguished by year of birth from age 19 till age 99. The absence of an upper limit on the size of shocks implies that households cannot hold negative amounts of equity. The annual cohorts option compares to the use of 9 cohorts by Krueger and Kubler (2006), four cohorts by Sánchez-Marcos and Sánchez-Martín (2006) and three cohorts by Brooks (2000). To avoid the curse of dimensionality that would block the use of a model with 81 cohorts, I use state space aggregation (Bertsekas and Castañon (1989)). That is, households use only the information from a few cohort aggregates to forecast next period's rates of return.

The advantage of distinguishing households on an annual basis is twofold. First, pension reform measures are usually defined on annual cohorts (or even monthly cohorts). Ten-year cohorts therefore constitute a rather coarse grid for the study of the effects of policy reform. Secondly, and perhaps more importantly, a discrete time model with e.g. ten-year time intervals implies that households are allowed to trade assets only once every decade. This constitutes a huge market incompleteness, that tends to overstate the amount of undiversifiable risk that households face. While an annual model is not equivalent to continuous trade either, it does approximate this setting better than models that use a coarser time base.

The remainder of this paper is subdivided as follows: Section 2 discusses the model, first the model of the firm and the stochastic return process on capital in Section 2.1, then the household model in Section 2.2, the PAYG pension scheme in Section 2.3, the government closure rule in Section 2.4 and finally the equilibrium conditions in Section 2.5. Issues in asset valuation in incomplete markets are discussed separately in Section 2.6. Results are discussed in Section 3, first the single-asset case in Section 3.1, then the effects of introducing a bond market in Section 3.2 and next the effects of a number of social security reforms in Section 3.3. Section 4 evaluates the results.

2 The Model

2.1 Firms

Firms mainly serve as a source of risk factors, related to the return on investment and human capital. As a consequence. the firm model contains no dynamic elements, with the exception of an adjustment delay of one period between investment and productive capacity. In addition, I assume that investment expenditures are deductible before taxes according to economic depreciation. This avoids introducing depreciation rights as a state variable.

The production function is

$$
\begin{aligned}
Y_t &= F[K_t, \zeta_L L_t] && (1)\\
&= \left[(\zeta_K K_t)^{1-1/\sigma_y} + (\zeta_{L_t} L_t)^{1-1/\sigma_y}\right]^{\frac{1}{1-1/\sigma_y}} \\
L_t &= \sum_\tau h_{t-\tau} L_{t,\tau} && (2)
\end{aligned}
$$

Effective employment is a productivity-weighted aggregate of employment of different age cohorts $L_{t,\tau}$,with age-specific productivity $h_{t-\tau}$. Productivity shocks occur in ζ_L. The value of ζ_L is known at the beginning of period t. Positive productivity shocks can be thought of as "process innovations" that reduce production costs. The distribution of ζ_L is assumed to be trend-stationary, so that the technology uncertainty is limited to movements around a trend. That is, technology shocks do not create permanent cost advantages. Another important source of uncertainty for entrepreneurial activity is product innovation, that can quickly depreciate existing activities and capital. In this paper, I take a reduced-form approach to this type of uncertainty and assume that valuation shocks occur in the rate of depreciation of capital, δ (see also Bohn (1999a)).

The dynamics are specified as

$$
\begin{aligned}
K_{t+1} &= \mathrm{e}^{-\delta_{t+1}}(K_t + I_t) && (3)\\
\ln \zeta_{L_{t+1}} \mathrm{e}^{-\psi(t+1)} &= \lambda_L \ln \bar{\zeta}_L + (1-\lambda_L) \ln \zeta_{L_t} \mathrm{e}^{-\psi t} + \varepsilon_{L_{t+1}} && (4)\\
\delta_{t+1} &= \bar{\delta} + \varepsilon_{\delta_{t+1}} && (5)
\end{aligned}
$$

Production possibilities are characterized by the state variables K_t and ζ_{L_t}. Labor productivity have the mean reversion property, and moves around a deterministic trend $\bar{\zeta}_L \mathrm{e}^{\psi t}$. The random variables ε_L and ε_δ are i.i.d. normal variates.

Investment is financed from internal funds E and share issues VN. If the flow of internal funds is sufficient to finance investment, the residual is paid out as dividends (DIV) and no new shares are issued. If the flow of internal funds falls short of investment, dividends are cut to zero and the firm issues new shares. It is assumed that depreciation rights D are equal to current investment.[2]

$$E_t = pY_t - p_{l_t} L_t \tag{6}$$

$$D_t = \left(1 - \mathrm{e}^{-\delta_t}\right) K_t \tag{7}$$

$$DIV_t = \max\left[E_t - I_t, 0\right] \tag{8}$$

$$VN_t = I_t - E + DIV_t \tag{9}$$

At the start of period t, the firm has $n_{v_{t-1}}$ shares outstanding. The market price per share is denoted p_{V_t}, and the market value of the firm is $V_t = p_{v_t} n_{v_{t-1}}$. The firm then issues $n_{v_t} - n_{v_{t-1}}$ new shares.[3] These n_{v_t} shares are traded *cum dividend*, i.e. with the dividend falling to the buyer.[4] The return r_k to equity n_{v_t} is therefore given by

$$1 + r_{k_{t+1}} = \frac{p_{v_{t+1}}}{p_{v_t} - DIV_t / n_{v_t}} \quad \Leftrightarrow$$

$$1 + r_{k_{t+1}} = \frac{V_{t+1}}{V_t - DIV_t + VN_t} \tag{10}$$

where $VN_t = p_{v_t}\left(n_{v_t} - n_{v_{t-1}}\right)$ denotes the value of new share issues by the firm. It is assumed that dividend payments are not taxed. It follows from (8),(9), and (10) that the return to shareholders does not depend on the financial policy of the firm. I normalize the number of shares to $n_v = 1$. $r_{k_{t+1}}$ is stochastic, as the market value of the firm in period $t+1$ depends both on the depreciation rate δ_{t+1} and labor productivity $\zeta_{L_{t+1}}$, which are not revealed until the beginning of period $t+1$. Section 2.6 discusses how r_k relates to the preferences of households, as the owners of the firm.

2.1.1 Optimum

The *state* of the firm is characterized by the available capital stock, the state of the technology (δ_t, ζ_{L_t}), and other variables outside of the control of the firm, represented by Ω_t.[5] Firms maximize the present value of their cash flow, given by

$$V(K_t, \zeta_{L_t}, \Omega_t) = \max_{I,L} \mathrm{E}\left[\sum\nolimits_{\tau=t}^{\infty} (DIV_t - VN_t) \prod\nolimits_{s=t+1}^{\tau} m_s^f \,\middle|\, K_t, \zeta_{L_t}, \Omega_t\right] \tag{11}$$

where the m_s^f denote the (stochastic) discount factor of future returns, to be discussed below. The expectation is conditional on the state of the firm at time t, so that the present value function

[2] This assumption avoids the introduction of yet another state variable, depreciation rights.

[3] The number of new shares issued is known at the start of period t, when the price p_{v_t} of shares is determined.

[4] Alternatively, if trades are ex dividend, the original owner decides about production and investment in the current period.

[5] A complete list of state variables will be provided in Appendix C.1.

may be written as $V_t = V(K_t, \zeta_{L_t}, \Omega_t)$. Substituting (3) in the right-hand side of (11), the first-order equations wrt. I and L are obtained [6]

$$\mathrm{E}_t \left[m^f_{t+1} \left(1 + r_{k_{t+1}}\right) \right] = 1 \tag{12}$$

$$\frac{\partial F[K_t, \zeta_{L_t} L_t]}{\partial L_t} = p_{l_t} \tag{13}$$

where the uncertain return to capital, r_{k_t}, can be written as

$$1 + r_{k_{t+1}} = \left(1 + \frac{\partial F[K_{t+1}, \zeta_{L_{t+1}} L_{t+1}]}{\partial K_{t+1}}\right) \mathrm{e}^{-\delta_{t+1}} \tag{14}$$

The return to capital depends on both risk factors, the depreciation rate δ_{t+1} and labour productivity $\zeta_{L_{t+1}}$. According to (14), the investment decision I_t also affects the distribution of returns in period $t+1$. Given the discount factor of investors, this suffices to determine the optimal amount of investment. However, in general the investment decision changes the discount factor of investors as well, so that (12) reflects both supply and demand considerations.

It is proved in Appendix A that the *ex dividend* market value of the firm equals the replacement value of the new capital stock

$$V(K_t, \delta_t, \zeta_{L_t}, \Omega_t) - DIV_t = K_t + I_t \tag{15}$$

2.2 Households

2.2.1 Utility

Households are divided into generations, distinguished by their year of birth t_0. The death hazard λ of a household depends on its age, $\lambda = \lambda_{t-t_0}$. In each generation, there is a continuum of households, so that the survival distribution of each cohort is deterministic, $\Lambda_{t-t_0+1} = (1-\lambda_{t-t_0})\Lambda_{t-t_0}$, where $\Lambda_0 = 1$. Each household maximizes expected lifetime utility, given by a non-expected utility formulation

$$\Upsilon_{t,t_0} = \left[u(c_{t,t_0}, l_{t,t_0})^{1-1/\gamma} + \frac{1-\lambda_{t-t_0}}{1+\rho} \left(\tilde{\Upsilon}_{t+1,t_0}\right)^{1-1/\gamma} \right]^{1/(1-1/\gamma)} \tag{16a}$$

$$\tilde{\Upsilon}_{t+1,t_0} = \mathrm{E}_t \left[\Upsilon^{\alpha}_{t+1,t_0} \right]^{1/\alpha} \tag{16b}$$

$\tilde{\Upsilon}_{t+1,t_0}$ is a "certainty-equivalent" utility measure, used by households to compare uncertain future utility with current consumption of goods and leisure (Epstein and Zin (1989)). $1-\alpha$ is the Arrow-Pratt coefficient of relative risk aversion. If $\alpha = 1$, households are risk neutral and only care about the distribution of consumption between periods, as specified by the intertemporal elasticity of substitution γ and the time preference parameter ρ. If $1-\alpha = 1/\gamma$,

[6] See Appendix A for derivations.

the risk aversion of households equals their preference for consumption smoothing and we obtain $(\Upsilon_{t,t_0})^{1-1/\gamma} = u_{t,t_0}^{1-1/\gamma} + \frac{1-\lambda_{\tau-t_0}}{1+\rho} \mathrm{E}\left[(\Upsilon_{t+1,t_0})^{1-1/\gamma} \right]$, which is an expected utility formulation (in terms of $U = \Upsilon^{1-1/\gamma}$). This parameter choice represents the "standard" specification of intertemporal choice, where no distinction is made between risk aversion and intertemporal consumption smoothing.

The subutility function u is characterised by perfect substitution between consumption of goods and a transformation of leisure

$$u(c_{\tau,t_0}, l_{\tau,t_0}) = c_{\tau,t_0} + \xi_{\tau,t_0} \frac{l_{\tau,t_0}^{1-\theta}}{1-\theta} - c_{\min_\tau} \tag{17a}$$

$$c_{\min_\tau} = \xi_{\tau,t_0} \frac{l_{\max}^{1-\theta}}{1-\theta} \tag{17b}$$

We assume that $\theta > 0$. The leisure preference parameters ξ_{τ,t_0} generally depend both on time τ, and birth cohort t_0. The inclusion of minimal consumption $c_{\min}$ prevents negative subutility. As a result, $c_{t,t_0} = u_{t,t_0} - \xi_{t,t_0} l_{t,t_0}^{1-\theta} / (1-\theta) + c_{\min_t} \geq \xi_{t,t_0} \left(l_{\max}^{1-\theta} - l_{t,t_0}^{1-\theta} \right) / (1-\theta) \geq 0$.

For analytic convenience, I reformulate the utility function (16) by using the transform $U_{t,t_0} = \Upsilon_{t,t_0}^{1-1/\gamma} / (1-1/\gamma)$.

$$U_{\tau,t_0} = \frac{u(c_{\tau,t_0}, l_{\tau,t_0})^{1-1/\gamma}}{1-1/\gamma} + \frac{1-\lambda_{\tau-t_0}}{1+\rho} \frac{\mathrm{E}_\tau \left[\left((1-1/\gamma)\, U_{\tau+1,t_0} \right)^{\alpha/(1-1/\gamma)} \right]^{(1-1/\gamma)/\alpha}}{1-1/\gamma} \tag{18}$$

2.2.2 Income and Wealth

At the start of period t the financial assets of a household are equity shares $n_{v_{t-1}}$, and bonds B_{t-1}. The household can trade its equity shares at the price p_{v_t}, which is determined at the opening time of markets in period t. Interest on bonds, $r_{b_{t-1}} B_{t-1}$, is paid at the start of period t.[7] Financial wealth at the start of period t is therefore

$$A_{t,t_0} = p_{v_t} n_{v_{t-1,t_0}} + \left(1 + r_{b_{t-1}} \right) B_{t-1,t_0} \tag{19}$$

For an individual household, the state vector contains its private wealth, A_{t,t_0}, its age $a = t - t_0$, and macro-economic variables summarized in Ω_t (see (32)). The only element of the state vector under the control of the household is A_{t,t_0}. The full household state vector is (A_{t,t_0}, a, Ω_t).

The government levies a a labor income tax τ_l on wage income, retirement income, and transfers, and a consumption tax τ_c on private consumption. Pension premiums are tax exempt. Taxes are linear and may vary with the state of the economy and with the age of the household. Households receive a transfer T_t from the government, that depends on age and possibly also on the state of the economy. During the retirement period, public pensions yield an income y_{P_t}

$$y_{P_{t,t_0}} = \omega_t \left(1 - \delta_{P_{t-t_0}} \right) \bar{p}_{l_t} \tag{20}$$

[7] Note that the dividend on the $n_{v_{t-1}}$ shares is collected in period $t-1$, as the shares are traded cum dividend.

where ω denotes the replacement rate of the pension fund, δ_{P_τ} is the eligibility indicator, which depends on age τ, and $\bar{p}_{l_t}$ is the average wage in period t.

The household can use its resources to buy consumption goods and financial assets. The cash on hand available for investment in financial assets in period t is

$$\begin{aligned} A^+_{t,t_0} = A_{t,t_0} + (1-\tau_{l_t}) \left(\left(1 - \delta_{P_{t-t_0}} \pi_{P_t}\right) p_{l_{t,t_0}} (l_{\max} - l_{t,t_0}) + T_{t,t_0} + y_{P_{t,t_0}} \right) \\ - (1+\tau_{c_t})) \, c_{t,t_0} \end{aligned} \tag{21}$$

c denotes consumption of goods and services, l is consumption of leisure, T represents the transfers from the government to households, τ_l is the income tax and τ_c denotes the consumption tax. π_P is the contribution rate to the pension fund, which is levied only during the pre-retirement period. The household supplies $l_{\max} - l$ units of labor per period.

The household invests an amount an amount B_{t,t_0} in bonds, and the remainder in equity. Since equity is bought *cum dividend*, the total value of the shares is $p_{v_t} n_{v_{t,t_0}} = A^+_{t,t_0} - B_{t,t_0} + n_{v_{t,t_0}} div_{t,t_0}$. The number of shares bought is then $n_{v_{t,t_0}} = \left(A^+_{t,t_0} - B_{t,t_0}\right) / \left(p_{v_t} - div_{t,t_0}\right)$ shares. To deal with the possibility that it does not survive till period $t+1$, the household sells claims to its remaining assets to other households, conditional on its death, as in Yaari (1965). The dynamic budget constraint is therefore[8]

$$\begin{aligned} (1-\lambda_{t-t_0}) A_{t+1,t_0} &= p_{v_{t+1}} n_{v_{t,t_0}} + (1+r_{b_t}) B_{t,t_0} \qquad (22) \\ &= \frac{p_{v_{t+1}}}{p_{v_t} - div_{t,t_0}} \left(A^+_{t,t_0} - B_{t,t_0}\right) + (1+r_{b_t}) B_{t,t_0} \quad \Rightarrow \\ (1-\lambda_{t-t_0}) A_{t+1,t_0} &= \left(1 + r_{k_{t+1}}\right) A^+_{t,t_0} + \left(r_{b_t} - r_{k_{t+1}}\right) B_{t,t_0} \qquad (23) \end{aligned}$$

where r_k is defined in (10).

2.2.3 Optimum

Utility maximization is subject to the budget constraint (23) and a time constraint

$$l_{t,t_0} \leq l_{\max} \tag{24}$$

The budget equation (23) depends on the characteristics of the individual household, (A_t, a_t), and on macroeconomic variables like factor prices, taxes, and labor productivity shocks. Maximum utility U can be written as a function of the state vector, $U = U(A_t, s_t)$, where $s_t = (a_t, \Omega_t)$

[8] We can rewrite the budget constraint to explicitly include all sources of capital income by writing (22) as

$$A_{t+1,t_0} = A^+_{t,t_0} + \underbrace{\left(p_{v_{t+1}} - p_{v_t}\right) n_{v_{t,t_0}}}_{\text{capital gain}} + \underbrace{n_{v_{t,t_0}} div_{t,t_0}}_{\text{dividend income}} + r_{b_{t+1}} B_{t,t_0}$$

are the state variables not under the control of the household. U is defined recursively as

$$U_t\left(A_{t,t_0}, s_t\right) = \max_{c_t, l_t, B_t} \frac{u(c_{t,t_0}, l_{t,t_0})^{1-1/\gamma}}{1-1/\gamma} + \frac{1-\lambda_{t-t_0}}{1+\rho} \frac{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha}}{1-1/\gamma} \tag{25}$$

Appendix B derives the first-order equations of the household decision problem (25).

Given the household value function $U\left(A_t, s_t\right)$, the demand equations for consumption and leisure follow

$$u_{t,t_0} = \left((1+\tau_{c_t})U_{A_t}\right)^{-\gamma} \tag{26a}$$

$$l_{t,t_0} = \left(\frac{1}{\xi_t} \frac{p_{l_{t,t_0}}\left(1+\lambda_{l_{t,t_0}}\right)(1-\tau_{l_t})\left(1-\delta_{P_{t-t_0}}\pi_{P_t}\right)}{1+\tau_{c_t}}\right)^{-1/\theta} \tag{26b}$$

$$c_{t,t_0} = u_{t,t_0} - \xi_{t,t_0} l_{t,t_0}^{1-\theta} / (1-\theta) + c_{\min_t} \tag{26c}$$

where $\lambda_{l_{t,t_0}}$ denotes the Lagrange multiplier constraint of leisure. Equation (26a) shows that there is a direct relation between the marginal utility of wealth and full consumption. Full consumption u_{t,t_0} has a spot price $1+\tau_{c_t}$. Instead of consuming now, the household may also save for future consumption, which yields a marginal utility U_A that is substituted against current consumption at an elasticity γ. Demand for leisure l depends only on the current real after-tax wage, as intertemporal substitution in leisure is assumed zero in the utility function (17a).

Saving and Portfolio Choice Next to the saving-consumption decision, the household must also decide which assets to invest its savings in. Appendix B derives a compact formulation for this decision by defining the *stochastic discount factor*

$$m_{t+1,t_0} = \frac{1}{1+\rho} \frac{U_{A_{t+1}}}{U_{A_t}} \left(\frac{(1-1/\gamma)\,U_{t+1,t_0}}{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\frac{\alpha}{1-1/\gamma}}\right]^{\frac{1-1/\gamma}{\alpha}}} \right)^{\frac{\alpha}{1-1/\gamma}-1} \tag{27}$$

The stochastic discount factor measures the value of a unit of wealth next period per unit of current wealth. It consists of three parts. The first fraction on the right-hand side of (27) captures the horizon of the household in terms of its impatience ρ. An impatient household saves less. The second fraction considers the marginal value of wealth in the next period per unit of value of current wealth, net of taxes. A household with a higher marginal value of current wealth saves less, as current euros are more "expensive" that future euros in terms of marginal utility yield. The last term, in brackets, compares next-period utility (conditional on survival) with its certainty-equivalent counterpart. A household that is *relatively* risk-averse, in the sense that $\alpha/(1-1/\gamma)>1$, has a certainty-equivalent utility that is lower than expected utility. So, for most states, the household applies a correction factor smaller than unity to next period's marginal utility, implying that it tends to discounts the future more heavily than would follow from the *ex post* ratio of marginal utilities.[9] That is, for any given return distribution the household will save *less*, i.e. it will require a higher risk premium, if the stated condition is satisfied. Intuitively, for $\alpha/(1-1/\gamma)>1$, consumption smoothing is valued less than risk reduction.[10]

The asset demand equations for bonds and equity can be written as

$$\mathrm{E}_t\left[m_{t+1,t_0}\left(1+r_{b_t}\right)\right]=1 \tag{28a}$$

$$\mathrm{E}_t\left[m_{t+1,t_0}\left(1+r_{k_{t+1}}\right)\right]=1-\lambda_{I_{t,t_0}} \tag{28b}$$

$$\lambda_{I_{t,t_0}} I_{t,t_0}=0 \tag{28c}$$

$$0\leq\lambda_{I_{t,t_0}}<1 \tag{28d}$$

As a result of the discrete nature of the decision process in this model, the optimal investment in equity must be nonnegative. Negative investment in equity runs the risk that the amount borrowed cannot be repaid with interest, if the return on investment is sufficiently high.[11] This implies that households will refrain from using the equity market, rather than financing debt by issuing equity, to avoid becoming insolvent. Given the parameterization of the model, this condition will indeed bind for young households, because households are rather impatient, young households have an increasing wage profile, and the returns to equity and wages are strongly correlated. [12] The net result of this restriction is a boost of the equity premium, as young households are excluded from the equity market.

As bonds are risk-free, we observe that the expected stochastic discount factor must satisfy

$$\mathrm{E}_t\left[m_{t+1,t_0}\right]=\frac{1}{1+r_{b_t}} \tag{29}$$

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[9]For $\gamma<1$, $\alpha<1-1/\gamma\Rightarrow\frac{\alpha}{1-1/\gamma}>1$. Then, by Jensen's inequality, $\mathrm{E}_t\left[\left(\left(1-\frac{1}{\gamma}\right)U_{t+1}\right)^{\frac{\alpha}{1-1/\gamma}}\right]^{\frac{1-1/\gamma}{\alpha}}>\mathrm{E}_t\left[\left(1-\frac{1}{\gamma}\right)U_{t+1}\right]$. For $\gamma>1$, both inequalities are reversed, so that the conclusion wrt. (27) still holds.

[10]In other words, consumption growth is not a sufficient statistic for the stochastic discount factor. Note that this does not deny the existence of precautionary saving. Precautionary saving occurs because of hedging behaviour to guard against large increases in marginal utility of wealth. This requires that marginal utility is concave in wealth (Carroll and Samwick (1998)).

[11] This is a difference with a continuous-time model, if the return process is normal. However, a continuous-time process with Poisson jumps in asset prices is similar to a discrete-time model.

[12] Note that this condition is different from the "junior can't borrow" argument in Constantinides et al. (2002), where households would like to hold positive equity, financed by issuing bonds, but cannot do so due to capital market imperfections.

If a riskless asset exists, (29) shows that the expected stochastic discount rate of all households must be the same.[13] A high degree of relative risk aversion lowers the risk-free rate. (29) allows us to define the riskless rate also in the absence of a risk-free asset, but in that case it will generally differ between generations.[14]

2.3 Pensions

The budget restriction of the PAYG pension scheme is given as

$$\sum_{t_0=-\infty}^{t} y_{P_{t,t_0}} N_{t,t_0} \Lambda_{t-t_0} = \sum_{t_0=-\infty}^{t} \delta_{P_{t-t_0}} \pi_{P_t} p_{l_{t,t_0}} \left(l_{\max} - l_{t,t_0}\right) + T_{P_t}$$

where T_{P_t} denote government transfers to the scheme. The closure rule depends on whether the scheme is DB or DC.

$$\text{Defined Contribution} \qquad \omega_t = \frac{\pi_P \sum_{t_0=-\infty}^{t} \delta_{P_{t-t_0}} p_{l_{t,t_0}} \left(l_{\max} - l_{t,t_0}\right) + T_{P_t}}{\bar{p}_{l_t} \sum_{t_0=-\infty}^{t} \left(1 - \delta_{P_{t-t_0}}\right) N_{t,t_0} \Lambda_{t-t_0}}$$

$$\text{Defined Benefit} \qquad \pi_{P_t} = \frac{\sum_{t_0=-\infty}^{t} y_{P_{t,t_0}} N_{t,t_0} \Lambda_{t-t_0} - T_{P_t}}{\sum_{t_0=-\infty}^{t} \delta_{P_{t-t_0}} p_{l_{t,t_0}} \left(l_{\max} - l_{t,t_0}\right)}$$

The government can use transfers T_{p_t} to stabilize the contribution rate in a DB system, or the replacement rate in a DC system. These transfers require tax changes, that may change the distribution of the tax burden over current and future generations, depending on the debt policy pursued by the government.

2.4 The Government

The dynamic budget restriction for the government is

$$B_{t+1} = \left(1 + r_{b_t}\right)\left(B_t + T_t + T_{P_t} - \tau_{c_t} c_t - \tau_{l_t} \sum_{\tau=-\infty}^{t} p_{l_{t,\tau}} \left(1 - \pi_{P_{t,\tau}} \left(l_{\max} - l_{t,\tau}\right)\right)\right) \tag{30}$$

where B denotes the value of government bonds and r_{b_t} the bond interest rate. The no-Ponzi game condition requires that $\lim_{t\to\infty} B_t \prod_{\tau=1}^{t} \left(1 + r_b(\tau)\right)^{-1} = 0$. I assume that the government follows a balanced-budget policy ($B_{t+1} = B_t$).[15] Different tax instruments can be used to satisfy this constraint (e.g. τ_c, τ_l). The tax rate used to balance the budget will be a function of the state variables, and will therefore be stochastic.

[13]Government bonds do not offer a safe return in real terms. Campbell and Viceira (2005) show that the real long-term bond risk is of the same size as the long-term equity risk.

[14]In that case it is the rate of return at which the household wants to hold a zero amount of riskless assets.

[15]This keeps bonds out of the list of state variables.

2.5 Equilibrium

Market equilibrium is given by

$$L_{t,\tau} = N_{t,\tau}\Lambda_{t-\tau}\left(l_{\max} - l_{t,\tau}\right) \quad (\tau = -\infty, \ldots, t) \tag{31a}$$

$$L_t = \sum h_{t-\tau} L_{t,\tau} \tag{31b}$$

$$I_t = \sum_{\tau=-\infty}^{t} \left(A^{+}_{t,\tau} - B_{t,\tau}\right) \tag{31c}$$

$$Y_t = \sum_{\tau=-\infty}^{t} c_{t,\tau} + I_t \tag{31d}$$

$$B_{t+1} = B_t \tag{31e}$$

$$B_t = \sum_{\tau=-\infty}^{t} B_{t,t} \tag{31f}$$

$$A_t = V_t + B_t \tag{31g}$$

where $N_{t,\tau}\Lambda_{t-\tau}$ denotes the size of generation τ, V denotes equity holdings, and B denotes bond holdings. Labor market equilibrium is formulated in (31a). The labor market clears through wages, p_{l_t}, which affects the supply and demand of labor. (31d) gives the equilibrium condition on the goods market. The net supply of bonds to the private sector is zero, as the government follows a zero-debt policy. As different households have different desired portfolios, a bond market is viable all the same.

The vectors in the state space consist of the following elements

$$\Omega_t = \left(K_t, \zeta_{L_t}, \{A_{t,\tau}\}_{\tau=-\infty}^{t}, \{N_{t,\tau}\}_{\tau=-\infty}^{t}\right) \tag{32}$$

where n_T denotes the maximal age attainable (i.e., $\Lambda_\tau = 0$ for $\tau > n_T$).[16] The dimension of the state space is therefore $2n_T + 2$. Depending on the number of age groups, the state space can be quite large. In appendix D, I discuss ways to reduce the dimension of the state space. A limitation of this paper that will be maintained througout is that the population will be in steady state, so that the population composition is not part of the state space.

2.6 The Value of Income Claims

In this section I discuss how agents and markets value the income from different assets. We start with equity, i.e. claims to the dividend stream of the firm. Inserting (10) in (28b) and rewriting yields

$$V_t = \mathrm{E}_t\left[m_{t+1,t_0} V_{t+1}\right] + DIV_t - VN_t \quad \forall t_0 \tag{33}$$

[16]The size of government debt also enters the state vector, if the government does not maintain a balanced budget policy, see Section 2.4.

With complete markets, it holds that $m_{t,t_0} = m_t \ \forall t_0$. All risks can be traded, so households must value all risks in the same way and apply the same discount rate to the (risky) dividend stream of firms. In an incomplete market setting this is not necessarily the case. Matters can be considerably simplified however, if the dividend stream is contained in the market subspace, i.e. if *partial spanning* occurs (Magill and Quinzii (1996), p. 384). In the model of this paper, partial spanning of entrepreneurial risk is present for those households who are allowed to trade stock at the margin. For these households equation (12) shows that investors must attach the same present value to next period's market value of the capital stock. While the stochastic discount rate of different generations may correlate differently with depreciation risk or productivity risk, the impact on the market value of next period's capital stock is the same for all generations. However, generations that hold a zero amount of stock have a lower valuation of the firm's market value.

The situation is different with respect to pension claims. In the absence of complete markets, differences in valuation of pension claims between households are inevitable, as households cannot directly trade their implicit pension claims. With both labor productivity risk and depreciation risk present, income shocks cannot be fully insured with a portfolio that consists only of equity and a riskless asset. In that case, a PAYG pension linked to wages offers partial insurance to old-age income uncertainty. However, as households cannot take arbitrary positions in the implicit claim, different generations will value the claim differently. Within the context of the present model, opening a market of wage-linked bonds would restore market completeness, and at the same time obviate the need for a pension system. However, there are always macroeconomic risk factors that are not fully covered by an asset, e.g. demographic uncertainty, so that markets are always incomplete.

The implicit market value of human capital and pensions can be evaluated by means of the stochastic discount rate. According to (20), the household has an implicit claim on an income stream of $y_{P_{t,t_0}} = \omega_t \left(1 - \delta_{P_{t-t_0}}\right) \bar{p}_{l_t}$ via the pension system. Let the current value of the claim to the income stream $\left(y_{P_{t,t_0}}, y_{P_{t+1,t_0}}, \dots\right)$ be $A_{P_{t,t_0}}$. The (uncertain) return to the claim equals $1 + r_{P_{t,t_0}} = \frac{A_{P_{t+1,t_0}}}{A_{P_{t,t_0}} - y_{P_{t,t_0}}}$ and the arbitrage condition gives $\mathrm{E}\left[m_{t+1,t_0}\left(1 + r_{P_{t,t_0}}\right)\right] = 1 \Leftrightarrow$, so

$$A_{P_{t,t_0}} = y_{P_{t,t_0}} + \mathrm{E}\left[m_{t+1,t_0} A_{P_{t+1,t_0}}\right] \tag{34}$$

This is a private valuation in the sense that different households attach a different value to the same income stream, if the stream cannot be spanned in the market. Similarly, the private valuation of human capital of generation t_0 is given by the recursion

$$H_{t,t_0} = p_{l_{t,t_0}} l_{\max} + \mathrm{E}\left[m_{t+1,t_0} H_{t+1,t_0+1}\right]$$

3 Results

I investigate the effects of incomplete markets on economic performance in a number of steps. First, the model is calibrated and solved for the closed economy case where the only asset market present is equity.[17] In addition to claims on capital income, households have implicit claims on social security. This setting provides a relatively favourable environment for social security, as old households can save only via the stock market, which has a high risk profile, and social security has added value as a quasi-asset with a different risk profile. Second, I add a bond market that provides risk-free claims on next period consumption goods. This broadens the scope of households to provide for their old-age income through private saving. With these two private asset markets in place, I investigate the effect of two social security reform measures, abolishing (privatising) social security and a switch from a DB to a DC system.

3.1 Equity Market Only

The equity market case acts as the benchmark case for the simulations. In view of the considerably long-run inflation risk present in nominal bonds (Campbell and Viceira (2005)), only price-indexed bonds can be labelled as risk-free and these bonds are not common. A model without a risk-free asset may therefore serve as a better first approximation to the real-world asset market structure than a model with a risk-free asset. The parameterization of the model has been determined from an initial calibration of the household model, assuming that the firm sector and the government sector are in long-run equilibrium. The resulting parameter values are in Table 1.

Table 1: Key parameters and indicators†

s_l	α	θ	σ_y	γ	ρ	$l_{\max}$	λ_L	$\bar{\delta}$	σ_{ε_L}	$\sigma_{\varepsilon_\delta}$	$\rho_{\varepsilon_L\varepsilon_\delta}$
0.7	-4	5.0	0.5	0.5	0.0557	1	0.2	0.1	0.01	0.15	0.5

† Symbols are defined in Appendix E

Figures 1 and 2 give the life cycle profiles of consumption and leisure for the first and last years in the sample. These profiles show a plausible path for leisure, as a result of the calibration of labour participation coefficients on the Dutch labour market in 2005. There is some difference in the average consumption paths between the two years, but the main difference is with investment in fixed assets. In the initial year (2005), households start to invest in fixed assets at age 30, and 50 years later they wait till age 39. This difference can be traced to the lower equity premium in the later years of the sample period.

[17] Details of the solution procedure are given in Appendix C.1

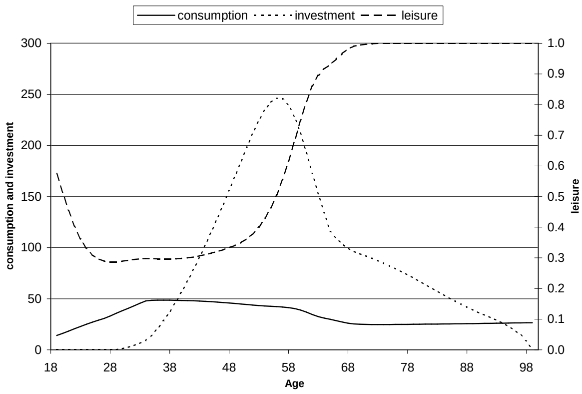

Figure 1: Average consumption, investment, and leisure profiles by age in period 1

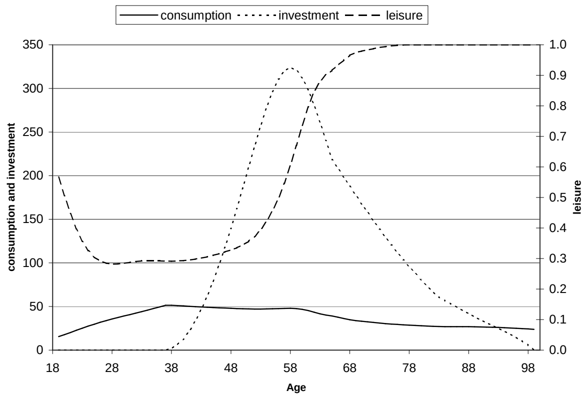

Figure 2: Average consumption, investment, and leisure profiles by age in period 50

The equilibrium solution of the model is a stochastic distribution. I present sample means and standard deviations of the long-run equilibrium distribution of a few variables in Table 2. The volatility of output and consumption are too high, whereas investment has approximately

Table 2: Model Statistics†

	y	C	I	L	p_L	r_k	$\mathrm{E}[r_k]-r_{b_{19}}$
mean	606	481	125	5.1	87	0.04	0.020
std.dev.	52	67	18	0.13	11	0.14	
	τ_c	ω	K	p_k	$\rho_{y,I}$	$\rho_{y,C}$	ρ_{y,p_L}
mean	0.18	0.099	1250	0.142	-0.84	0.98	0.995
std.dev.	0.003	0.002	350	0.055			

† Symbols are defined in Appendix E

the right volatily. The high volatility of output is a result of depreciation shocks to capital. Real wages are procyclical, in accordance with observations, but the correlation coefficient between wages and output is too high. The risk premium starts out at approximately 4.5%, but in the long run, at age 19, it is substantially lower at 2%. However, in this benchmark the equity premium is age-dependent. It increases again with age from aroung age 48. Figure 3 provides a graph of the distribution of the sample path of output. The process reaches a steady state after about twenty years. The residual variation in the sample mean is due to sampling variance (100 draws). Figure 4 gives the equity premium of a 64-year old worker. The variance of the process is highly nonlinear as a result of the assumed log-normality of the process.[18]

[18] If the mean of the process is m, and the variance is s^2, the parameters of the lognormal distribution are given by $\sigma^2 = \ln(1+s^2/m^2)$ and $\mu = \ln a - 0.5\,\sigma^2$. The displayed standard deviations are $m\exp[\pm\sigma]$.

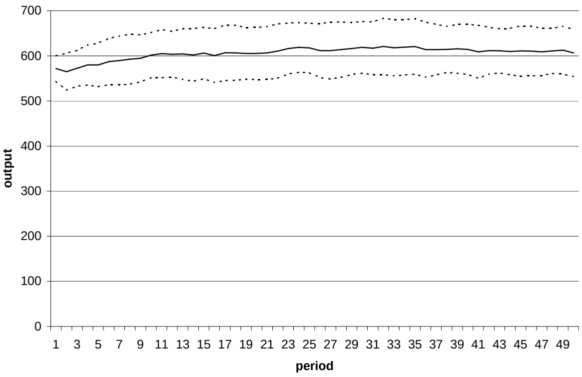

Figure 3: Mean output with one-sigma boundaries

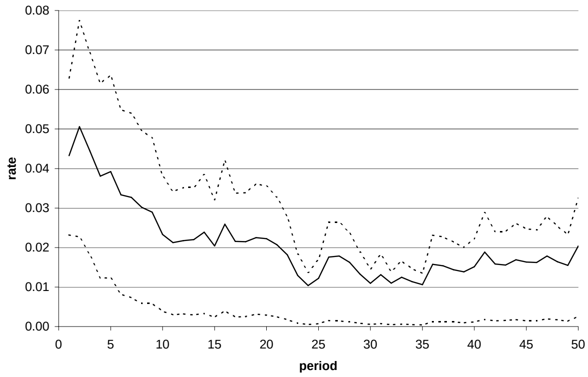

Figure 4: equity premium with one-sigma boundaries

3.2 Adding a Bond Market

A market for real bonds allows households to diversify their portfolio by age. Young households have a large amount of human capital, which provides a hedge against negative returns to equity. However, households are fairly impatient, with a time preference of 5.5%, and a wage profile that initially increases with age. Furthermore, the returns to equity and wages are strongly positively correlated. As a result, young households do not want to hold a positive position in equity. As they cannot hold negative amounts of equity, these households only have a negative position in bonds. Figure 5 gives the portfolio composition by age group for selected sample years. Households hold negative financial wealth until somewhere between age 30 and

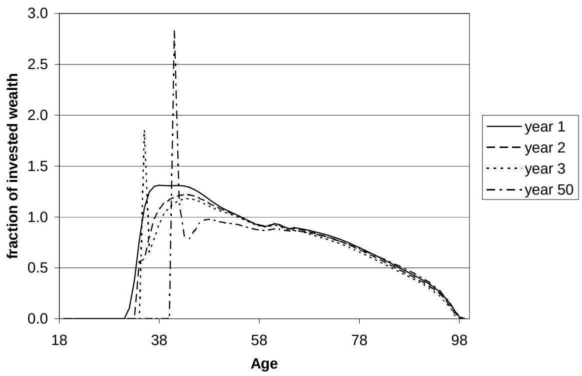

Figure 5: Mean equity shares in invested wealth by age for selected years

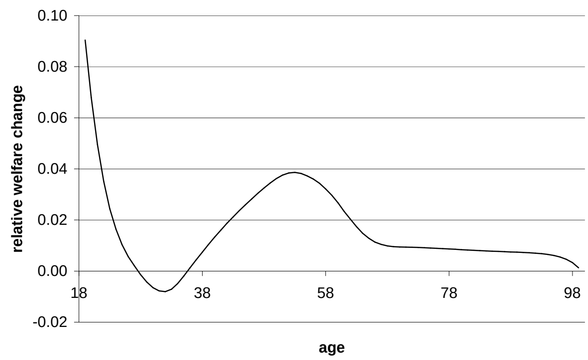

Figure 6: Welfare effects of the opening of a bond market

40, depending on the period under consideration. Between the ages 30-40 and 55 households hold more than 100% of their wealth in the form of equity. After age 55, households keep part of their wealth as bonds, and the fraction of financial wealth held as common stock gradually falls to zero.

Figure 6 shows that the welfare change, in terms of consumption gain, of introducing a bond market is positive for most generations. The opening of a bond market enables the young to take a negative position in bonds, and the old to invest part of their wealth in bonds. However, generations that have a net bond position of approximately zero after the opening of the bond market do not stand to gain much. In fact, a few generations experience a small fall in remaining lifetime utility, due to the fall in wages.[19]

The macroeconomic effects that correspond to these portfolio changes are depicted in Figures 7-8. The opening of a bond market does not boost growth. Young households, who previously held zero financial wealth, now can increase current consumption by borrowing against future income. This raises the equity premium and lowers the wage rate. This helps old generations, but is not particularly beneficial for generations that are just starting to invest in equity. Figure 7 depicts the decline in capital due to the opening of a bond market. The decline in capital is accompanied by a fall in after-tax wages, so that the decline in capital is reinforced by a fall in employment. Figure 8 presents the result.

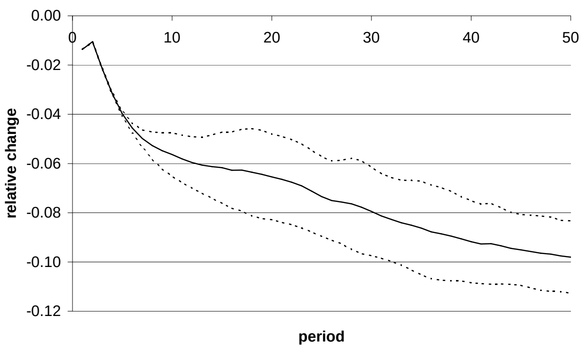

Figure 7: Capital response to the opening of a bond market with one-sigma boundaries

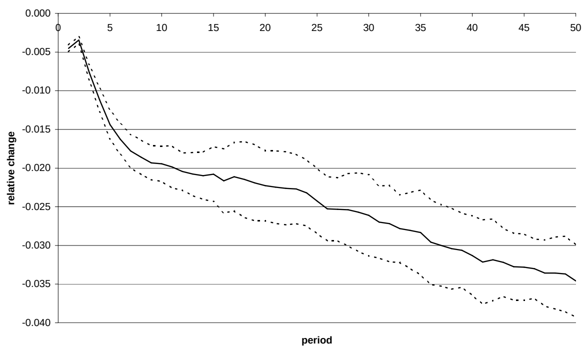

Figure 8: Output response to the opening of a bond market with one-sigma boundaries

Figure 9 gives the average return to bonds and the mean expected equity premium that result from the opening of the bond market. This graph shows that the average returns to bonds and equity are quite reasonable. The change in factor prices is presented in Figure 10 in terms of a fall in net wages. The before-tax fall in wages is somewhat larger, because PAYG benefits are linked to wages.

3.3 Social Security Reform

The shifting demographic composition towards elder citizens has generated considerable research into the scope for pension reform. Whereas in deterministic models social security only crowds out saving and distorts labour supply (see Lindbeck and Persson (2003)), in stochastic

[19] Note that markets are still not perfect after the opening of the bond market, because a) productivity risk is not insured and b) households cannot take a negative position in equity.

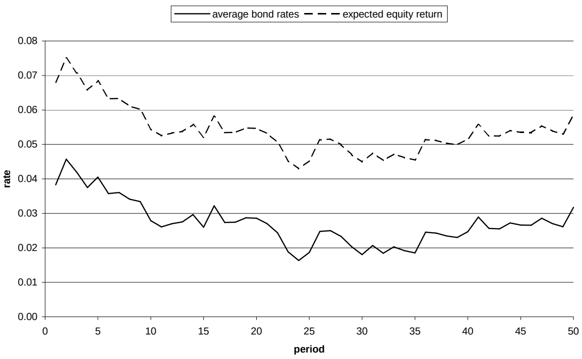

Figure 9: mean return to bonds and expected equity returns

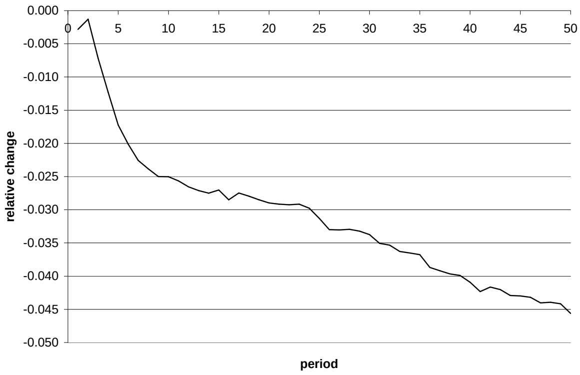

Figure 10: after-tax change in real wages

models the insurance aspects of social security makes for a more balanced story. Stochastic models allow for an assessment of the risk sharing aspects of social security vis-à-vis the distortions. Matsen and Thøgersen (2004) find that an optimal pension system depends partly on a paygo fund, and Bohn (2002) argues that the government can mimic an optimal pension system through debt management with a wage-indexed DB system.[20] In this section, I investigate two reform options, a switch from DB to DC, and a privatisation of social security.

3.3.1 A switch from DB to DC

The first option to consider is a switch from a Defined Benefit scheme to a Defined Contribution scheme. In a DB scheme, pensioners are relatively well-insured against the pension component of their old-age income. They are fully protected against shocks on the equity market. In a price-indexed DB scheme, pensioners are also fully protected against productivity shocks, so that the working-age populations bears all risks. In a wage indexed scheme, the imbalance is less severe, but it is still the case that workers bear most of the risk, sot that they are actively discouraged from supplying labour during periods with high contribution rates. A DC scheme with wage indexed pensions provides stable contribution rates, but at the cost of a lower insurance against all types of future shocks. The optimal insurance mix depends on the kind of shocks that one wants to insure against. E.g. Bohn (1999b) points out that a DC scheme is disadvantageous in terms of demographic risk, since it provides low benefits in times where capital returns are low due to small working-age cohorts. In this section I look specifically at the trade-off between both schemes in terms of productivity risk sharing.

Figure 11 shows the effects on the replacement rate and the contribution rate of the switch. The fixed DB premium is converted at an ex ante equal rate to a fixed DC contribution rate.

[20]Wage indexation implies a claim of pension benefits on human capital, which, in the absence of *endogenous* human capital accumulation, implies a paygo element.

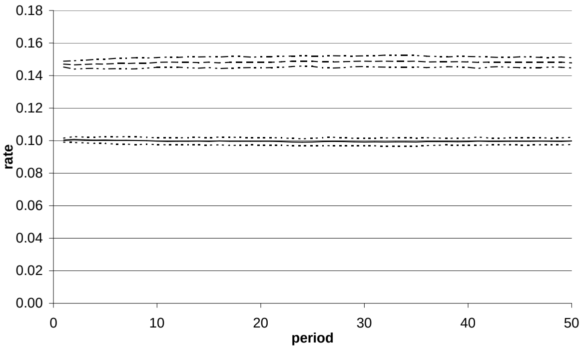

Figure 11: Time paths of contribution rates "—" and replacement rates "- - -" with one-sigma boundaries.

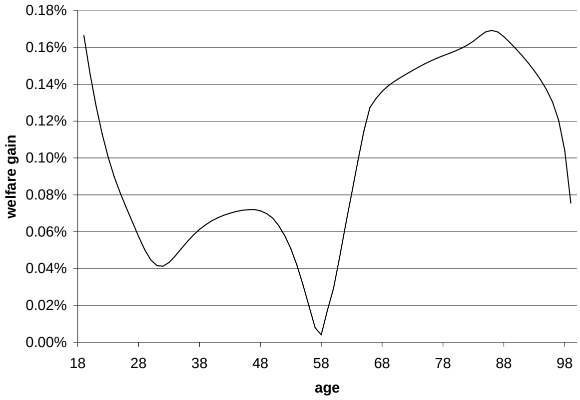

Figure 12: Welfare effects of a switch from DB to DC.

The graph shows that this ex ante equality is well maintained over the sample period. The replacement rates are somewhat more variable than the contribution rates, but variations in the contribution rates are distortionary, in contrast to replacement rate variations. Figure 12 shows that ex ante the conversion to a DC system is welfare-improving. The net gain of the conversion is fairly small, though, at about 0.1% of remaining lifetime consumption on average.

3.3.2 Privatising Social Security

A more drastic reform option is to privatise social security. In a privatised social security system, households pay mandatory contributions to private saving accounts. These contributions earn an actuarially fair rate of return on some asset market, chosen by the social security fund (or possibly the household itself). However, in the absence of liquidity constraints, households can easily adapt their portfolio to neutralize the actions of the social security fund. The net effect of a system of private saving accounts is then very similar to a setting without social security. In this section, I investigate the effects of privatising social security by lowering the replacement rate of social security by 10%. This reduction in social security is compensated for by a corresponding increase in government debt, that is used to reimburse existing generations for the expected loss in income. The compensation is calculated from equation (34) in section 2.6, i.e. using the proper stochastic discount rate.[21] Figure 13 specifies the value of the bonds to be handed over by the government to existing generations, per generation member.

The pension reform is ex ante neutral, but, in the absence of lump-sum taxation, ex post welfare effects will occur. As pointed out by many authors, e.g. Auerbach and Kotlikoff (1987);

[21] In the benchmark calibration, the total compensation needed to completely abolish PAYG social security is €539 milliard.

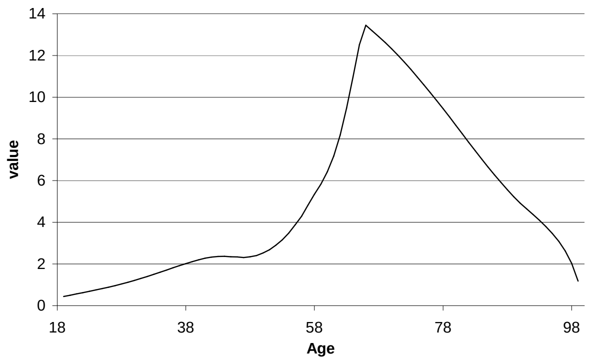

Figure 13: Compensation of existing generations for the loss in PAYG pensions

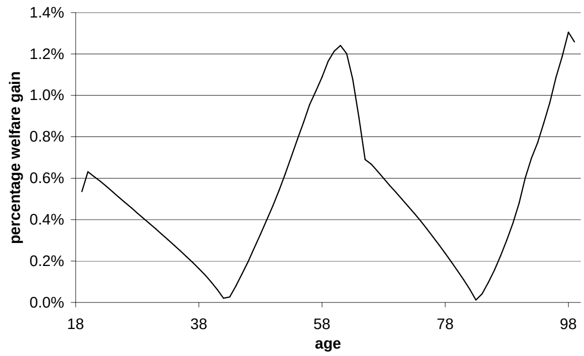

Figure 14: Welfare effects of privatising social security

de Nardi et al. (1999); Krueger and Kubler (2006), paygo social security crowds out private saving, and privatising social security generates a higher long-run capital stock. On the other hand, a privatised system no longer provides a proxy for a wage-indexed bond, and generates additional precautionary saving that hinders consumption smoothing over time. Honoring the implicit claims of current elderly lowers the crowding-in effects of privatization. A proper assessment of the welfare effects of privatizing social security needs to take into account these transition costs. Figure 14 gives the welfare effects of a shift to a privatised system in which the implicit claims of existing generations are acknowledged. All generations gain, but the welfare gains are largest for young generations and generations that are about to retire. Middle-aged generations gain less, because they invest heavily in stock, and the return to equity falls as the economy expands.

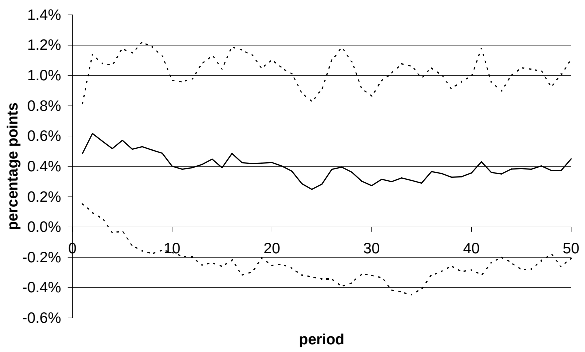

Figure 15: consumption tax effects of privatising social security

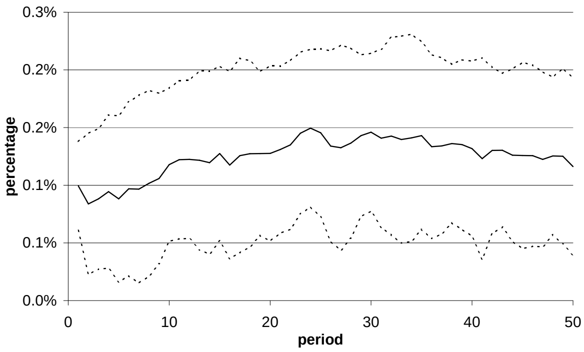

Figure 16: Labour supply effects of privatising social security with one-sigma boundaries

The macroeconomic effects of the privatization are given in Figures 15-18. The consumption tax increases by about 0.5%-point to pay for the increase in government debt. However, the simultaneous lowering of contribution rates boosts labour supply. Figure 16 and 18 show that privatisation boosts both labour supply and capital accumulation. The boost to labour supply leads to a small decline in capital returns. In addition, the equity premium falls, as young households need to pay less into the PAYG fund, and save the difference in terms of larger (less negative) bond holdings.

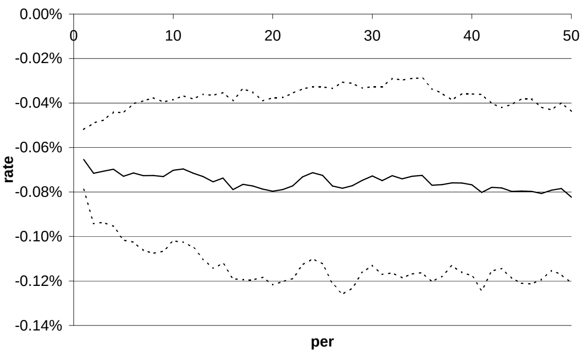

Figure 17: Equity premium effects of privatising social security

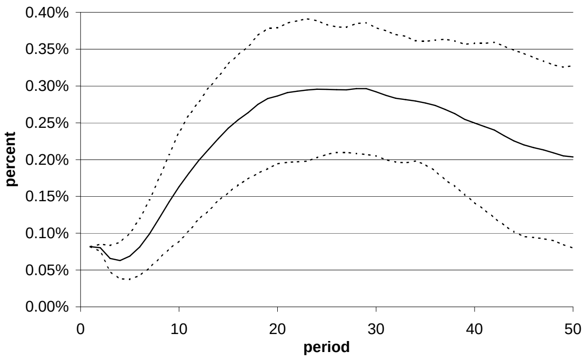

Figure 18: Capital supply effects of privatising social security with one-sigma boundaries

The long-run effects of privatizations on the equity premium show that the scope for self-insurance of households is so large, that the positive effects of creating a proxy for a wage-indexed bond through paygo social security are outweighed by the negative effects of the ensuing labour market distortion and the crowding out of private saving.

4 Conclusion

This paper uses a CGE-OLG model with macroeconomic risks to study the welfare effects of pension reform. The macroeconomic risks distinguished are investment return risk and labour productivity risk. In the absence of a market for productivity risk, public pensions that are linked to wages have an independent value added, as they offer an implicit asset that is not available in the market.

The paper offers a number of conclusions. First, the benefits of wage-indexed social security are outweighed by the distortions it generates with respect to labour supply and private saving. Reducing the size of social security leads to an increase in labour supply and saving and makes all generations better off. This shows that the paygo system investigated is not efficiency-enhancing, despite the lack of a market for wage-indexed bonds. Within the framework of this model, there are a number of reasons for this result. First, in comparison with the persistence in the risk factors investigated, households are long-lived, and in an ex ante sense are well able to insure against productivity risk and capital return risk over their entire life span. The extent of the additional insurance against productivity risk that is offered by the paygo fund is therefore limited. Moreover, the assumption of perfect annuity markets implies that the paygo system has no value added as a longevity insurance scheme. Finally, the long-term correlation between equity returns and wages is sufficiently high for the absent wage bonds market to make little difference in terms of economic efficiency.

Second, a switch from a defined-benefit system to a defined-contribution system does not offer any substantial advantages if the size of the pension system remains the same. The reduction in the volatility of contribution rates is only in the order of one percentage point, and must be weighed against the increase in the volatility of replacement rates.

To put these negative conclusions about the welfare effects of a paygo pension system into perspective, it is useful to point to a number of limitations of this study. First, it has not been shown that full privatisation of social security is Pareto-improving. The size of the pension system may simply be too large. A smaller pension may still offer a net welfare gain. Second, the pension scheme investigated here may be organized in a suboptimal way. In comparison with the opening of a wage-indexed bond market (which would necessarily be welfare-improving), two aspects come to mind: the absence of a linkage between contributions and benefits in the current scheme, and the uniform contribution rate paid by all participants. Indeed, in the presence of a wage-indexed bond market, it would be optimal for young workers to short wage indexed bonds, rather than accumulate them.

Appendices

A Firm Optimality

The value function is written as $V_t = V(K_t, \delta_t, \zeta_t, \Omega_t)$. Application of the maximum principle yields the following expression for the market value of the firm

$$V(K_t, \delta_t, \zeta_{L_t}, \Omega_t) = \max_{I,L} \mathrm{E}_t \left[m^f_{t+1} V(K_{t+1}, \delta_{t+1}, \zeta_{L_{t+1}}, \Omega_{t+1}) \right] + DIV_t - VN_t \tag{A.1}$$

where m^f_{t+1} is the (stochastic) discount factor of future returns applied by the owners of the firm. Substituting (3) in the right-hand side of (A.1), the first-order equations wrt. I and L are obtained

$$\mathrm{E}_t \left[m^f_{t+1} V_{K_{t+1}}(K_{t+1}, \delta_{t+1}, \zeta_{L_{t+1}}, \Omega_{t+1}) \mathrm{e}^{-\delta_{t+1}} \right] = 1 \tag{A.2}$$

$$\frac{\partial F[K_t, \zeta_{L_t} L_t]}{\partial L_t} = 1 \tag{A.3}$$

To find the Euler equation for K, we differentiate the value function (A.1) wrt. K_t.[22] Substi-

[22]Differentiability of the value function depends on the concavity and differentiability of the production function (see Benveniste and Scheinkman (1979)). In addition, the optimal solution must exist, for which we need to impose conditions on the discount rate m (see Section 2.6).

tute (3) for K_{t+1} and use the envelope theorem to find

$$V_{K_t}(K_t, \delta_t, \zeta_{L_t}, \Omega_t) = \mathrm{E}_t \left[m^f_{t+1} V_{K_{t+1}}(K_{t+1}, \delta_{t+1}, \zeta_{L_{t+1}}, \Omega_{t+1}) \mathrm{e}^{-\delta_{t+1}} \right] + F_{K_t} \tag{A.4}$$

Inserting (A.2) in (A.4) we obtain

$$V_{K_t}(K_t, \delta, \zeta_{L_t}, \Omega_t) = 1 + F_{K_t} \tag{A.5}$$

As the value function is obviously homogeneous of degree one in K_t, the market value of the firm can be written as $V(K_t, \delta_t, \zeta_{L_t}, \Omega_t) = V_{K_t} K_t$. The market value consists of the replacement value of the capital stock net of depreciation, and the capital share in production. The market value of the firm can be linked to the replacement cost of the capital stock by using (6)-(9) to write the dividend equation as

$$DIV_t = y_t - p_{l_t} L_t - I_t \tag{A.6}$$

combining this expression with (A.5) shows that the *ex dividend* market value of the firm equals the replacement value of the new capital stock

$$V(K_t, \delta_t, \zeta_{L_t}, \Omega_t) - DIV_t = K_t + I_t \tag{A.7}$$

Using (A.5) and (A.7), the (stochastic) return to equity in (10) can be written as

$$r_{k_{t+1}} = \left(1 + F_{K_{t+1}}\right) \mathrm{e}^{-\delta_{t+1}} - 1 \tag{A.8}$$

The return to equity depends on the difference between the marginal product of capital in the next period, which is a function of the rate of investment and the marginal product of labor, and the depreciation rate. This way both the depreciation rate and the productivity shock ζ_L affect the realized return to capital.

To find the investment equation we substitute (A.5) in (12) for time $t+1$, which gives

$$\mathrm{E}_t \left[m^f_{t+1} \mathrm{e}^{-\delta_{t+1}} \left(1 + F_{K_t}\right) \right] = 1 \quad \Leftrightarrow \tag{A.9a}$$

$$\mathrm{E}_t \left[m^f_{t+1} \left(1 + r_{k_{t+1}}\right) \right] = 1 \tag{A.9b}$$

In (A.9a), the marginal product of capital in period $t+1$, F_{K_t}, depends on labor supply in period $t+1$ or, equivalently, on the wage rate in $t+1$. The optimal investment decision therefore depends in general on the same state vector as household decisions (see Section 2.2). The formulation of the optimality condition for investment in (A.9b) relates to the discussion of the risk premium in Section 2.2.3.

B Household Optimality

The first-order equations for investment in equity, bonds, and leisure, conditional on being alive in period $t+1$ are found by differentiating (25) wrt. I, B, and l

$$\frac{u_{t,t_0}^{-1/\gamma}}{1+\tau_{c_t}} = \frac{1-\lambda_{t-t_0}}{1+\rho}\left\{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha-1}\right.$$
$$\left.\cdot\mathrm{E}_t\left[\left((1-1/\gamma)_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)-1} U_{A_{t+1}}\frac{1+r_{k_{t+1}}}{1-\lambda_{t-t_0}}\right]+\lambda_{I_{t,t_0}}\right\} \tag{B.1a}$$

$$\frac{u_{t,t_0}^{-1/\gamma}}{1+\tau_{c_t}} = \frac{1-\lambda_{t-t_0}}{1+\rho}\left\{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha-1}\right.$$
$$\left.\cdot\mathrm{E}_t\left[\left((1-1/\gamma)_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)-1} U_{A_{t+1}}\frac{1+r_{b_t}}{1-\lambda_{t-t_0}}\right]\right\} \tag{B.1b}$$

$$\xi_t l_t^{-\theta} = p_{l_{t,t_0}}\left(1-\tau_{l_t}\right)\left(1-\delta_{P_{t-t_0}}\pi_{P_t}\right)/\left(1+\tau_{c_t}\right) \tag{B.1c}$$

λ_I is the Lagrange multiplier for the constraint $I \geq 0$. To interpret (B.1a) and (B.1b), it is useful to derive the equations of motion of U_A. Differentiate (25) wrt. A_t, using (23) to obtain

$$U_{A_t} = \frac{1-\lambda_{t-t_0}}{1+\rho}\left\{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha-1}\right.$$
$$\left.\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\alpha/(1-1/\gamma)-1} U_{A_{t+1}}\frac{1+r_{k_{t+1}}}{1-\lambda_{t-t_0}}\right]\right\} \tag{B.2}$$

(B.2) allows us to simplify (B.1a) and (B.1b) by substituting for the expectations. These equations may be rewritten by dividing both sides by U_{A_t} and regrouping

$$\mathrm{E}\left[m_{t+1,t_0}\left(1+r_{k_{t+1}}\right)\right] \leq 1 \tag{B.3}$$
$$\mathrm{E}\left[m_{t+1},t_0\left(1+r_{b_t}\right)\right] = 1 \tag{B.4}$$

where m is the stochastic discount rate:

$$m_{t+1,t_0} = \left(\frac{(1-1/\gamma)\,U_{t+1,t_0}}{\mathrm{E}_t\left[\left((1-1/\gamma)\,U_{t+1,t_0}\right)^{\frac{\alpha}{1-1/\gamma}}\right]^{\frac{1-1/\gamma}{\alpha}}}\right)^{\frac{\alpha}{1-1/\gamma}-1}\frac{U_{A_{t+1}}}{U_{A_t}}\frac{1-\lambda_{t,t_0}}{1+\rho} \tag{B.5}$$

The stochastic discount rate allows for a completely symmetric formulation of the optimality conditions for investment in equity and bond.

C Model

The value function of a household of age a is given in (25) as $U_t\left(A_{t,t_0}, a, \Omega\right)$. The value functions are constructed recursively as

$$U\left(A_{t,a}, a, \Omega_t\right) = \max_{u_{t,a}} \left\{ \frac{u_{t,a}^{1-1/\gamma}}{1-1/\gamma} + \frac{1-\lambda_a}{1+\rho} \cdot \frac{\mathrm{E}_t\left[\left\{(1-1/\gamma)\, U\left(A_{t+1,a+1}, a+1, \Omega_{t+1}\right)\right\}^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha}}{1-1/\gamma} \right\}$$

To compute the expectation in this expression, households need to forecast the macro state Ω_{t+1}. This issue is addressed in Section C.1 below. *Given* the value functions for all cohorts, the model is

- Households

$$u_{t,a}^{-1/\gamma} = U_{A_{t,a}}(A_{t,a}, A_{P_{t-1,a}}, a; \Omega_t)/p_{u_{t,a}} \tag{C.1a}$$

$$u_{t,a}^{-1/\gamma} = \frac{1-\lambda_a}{1+\rho} \left\{ \mathrm{E}_t\left[\left((1-1/\gamma)\, U_{t+1,a}\right)^{\alpha/(1-1/\gamma)}\right]^{(1-1/\gamma)/\alpha-1} \mathrm{E}_t\left[\left((1-1/\gamma)\, U_{t+1,a}\right)^{\alpha/(1-1/\gamma)-1} U_{A_{t+1}}\left(1+r_{k_{t+1}}\right)\right] + \lambda_{I_{t,t_0}} \right\} \tag{C.1b}$$

$$0 = \mathrm{E}_t\left[\left((1-1/\gamma)\, U_{t+1,a}\right)^{\alpha/(1-1/\gamma)-1} U_{A_{t+1}}\left(r_{b_t} - r_{k_{t+1}}\right)\right] \tag{C.1c}$$

$$p_{u_{t,a}} = 1 + \tau_{c_t} \tag{C.1d}$$

$$\hat{p}_{l_{t,a}} = (1-\tau_{l_t})\,(1-\delta_{P_a}\pi_{P_t})\, p_{l_{t,a}} \tag{C.1e}$$

$$c_{t,a} = u_{t,a} - \xi_{t,a} l_{t,a}^{1-\theta}/(1-\theta) + c_{\min_t} \tag{C.1f}$$

$$l_{t,a} = \left(\frac{1}{\xi_t}\frac{\hat{p}_{l_{t,a}}}{1+\tau_{c_t}}\right)^{-1/\theta} \tag{C.1g}$$

$$y_{P_{t,a}} = \omega_t\,(1-\delta_{P_a})\,\bar{p}_{l_t} \tag{C.1h}$$

$$A_{t,a}^{+} = (1-\tau_A)A_{t,a} + (1-\tau_{l_t})\left(y_{P_{t,a}} + (1-\delta_{P_a}\pi_{P_t})\, p_{l_{t,a}} l_{\max}\right) - p_{u_{t,a}} u_{t,a} \tag{C.1i}$$

$$(1-\lambda_a)\,A_{t+1,a} = (1+r_{k_t})\left(A_{t,a}^{+} - B_{t,a}\right) + (1+r_{b_t})\,B_{t,a} \tag{C.1j}$$

$$A_{P_{t,a}} = \frac{p_{l_t}}{p_{l_{t-1}}} A_{P_{t-1,a}} + \delta_{P_a}\, p_{l_{t,a}}\,(l_{\max} - l_{t,a}) \tag{C.1k}$$

- Firms

$$L_t = \sum_{a=1}^{n_T} L_{t,a}\, h_{t,a} \tag{C.2a}$$

$$y_t = \left((\zeta_K K_t)^{1-1/\sigma_y} + (\zeta_{L_t} L_t)^{1-1/\sigma_y} \right)^{1/\left(1-1/\sigma_y\right)} \tag{C.2b}$$

$$p_{l_t} = \frac{\partial y_t}{\partial L_t} \tag{C.2c}$$

$$p_{l_{t,a}} = p_{l_t}\, h_{t,a} \tag{C.2d}$$

$$E_t = pY_t - p_{l_t} L_t \tag{C.2e}$$

$$D_t = \left(1 - \mathrm{e}^{-\delta_t}\right) K_{t-1} \tag{C.2f}$$

$$DIV_t = E_t - I_t \tag{C.2g}$$

$$V_t = (1 + F_{K_t})\, K_t \tag{C.2h}$$

$$r_{k_t} = (1 + F_{K_t})\, \mathrm{e}^{-\delta_t} - 1 \tag{C.2i}$$

- Government

$$B_t^s = B_t + \sum_{a=-\infty}^{t} \left(T_t - \tau_{c_t} c_{t,a} - \tau_{l_t} p_{l_t} \left(1 - \pi_{P_{t,a}}\right) (l_{\max} - l_{t,a}) \right) N_{t,a} \Lambda_{t-a} \tag{C.3}$$

- Equilibrium

$$y_t = \sum_{a=-\infty}^{t} c_{t,a} N_{t,a} \Lambda_{t-a} + I_t \tag{C.4a}$$

$$L_{t,a} = N_{t,a}\, (l_{\max} - l_{t,a}) \quad (a = -\infty, \ldots, t) \tag{C.4b}$$

$$A_t^{+} = I_t + K_t + B_t^s \tag{C.4c}$$

$$B_t = (1 + r_{b_t})\, B_t^s \tag{C.4d}$$

$$\sum_{t_0=-\infty}^{t} y_{P_{t,t_0}} N_{t,t_0} \Lambda_{t-t_0} = \pi_{P_t} \sum_{t_0=-\infty}^{t} \delta_{P_{t-t_0}}\, p_{l_{t,t_0}} \,(l_{\max} - l_{t,t_0}) \tag{C.4e}$$

- Dynamics

$$N_{t+1,a+1} = (1 - \lambda_a)\, N_{t,a} \tag{C.5a}$$

$$N_{t,1} = \sum_{i=1}^{n_T} \varphi_i N_{t-n_1,i} \qquad (\varphi_i \geq 0,\ i = 2, \ldots, n_T) \tag{C.5b}$$

$$K_{t+1} = (K_t + I_t)\, \mathrm{e}^{-\delta_{t+1}} \tag{C.5c}$$

$$\ln \delta_{t+1} = \ln \bar{\delta} + \varepsilon_{\delta_{t+1}} \tag{C.5d}$$

$$\ln \zeta_{L_{t+1}} \mathrm{e}^{-\psi(t+1)} = \lambda_L \ln \bar{\zeta}_L + (1 - \lambda_L) \ln \zeta_{L_t} \mathrm{e}^{-\psi t} + \varepsilon_{L_{t+1}} \tag{C.5e}$$

It appears that $\Omega_t = (K_t, \zeta_t, \{A_{t,a}\}_{a=1}^{n_T}, \{N_{t,a}\}_{a=1}^{n_T})$. With a large number of generations, the state space can become quite large. Appendix D discusses a way to reduce the dimension of the model without losing all information.

C.1 Solution Algorithm

The solution algorithm has the following steps

1. choose cohort aggregation matrix Γ_1, Γ_2 and define the cohort distributions $\bar{A} = \Gamma_1 \{A_{t,a}\}_{a=1}^{n_T}$, $\bar{N} = \Gamma_2 \{N_{t,a}\}_{a=1}^{n_T}$. Define the cohort state vector $\bar{\Omega}_t = (K_t, \zeta_{L_t}, \bar{A}_t, \bar{N}_t)$.

2. Define a grid $\mathscr{O}$ on the cohort state space.

3. Choose an initial mapping from $\bar{\Omega}$ to prices, $(r_k(\bar{\omega}), p_l(\bar{\omega}), \tau_c(\bar{\omega}), \pi(\bar{\omega}))$.

4. Construct the sequence of value functions $U(A_{t,a}, a, \Omega_t)$ by solving the recursion

$$U\left(A_{t,a}, a, \bar{\Omega}_t\right) = \max_{u_{t,a}} \left\{ \frac{u_{t,a}^{1-1/\gamma}}{1-1/\gamma} \right.$$
$$\left. + \frac{1-\lambda_a}{1+\rho} \frac{\mathrm{E}_t \left[\left\{ (1-1/\gamma) U\left(A_{t+1,a}, a, \bar{\Omega}_{t+1}\right) \right\}^{\alpha/(1-1/\gamma)} \right]^{(1-1/\gamma)/\alpha}}{1-1/\gamma} \right\}$$

   on the grid $(A_{t,a}) \times \mathscr{O}$. Note that the value functions are constructed for all ages, not just for the cohorts. The expanded grid includes the *individual* asset levels.

   Solve the model (C.2)-(C.4) for given value functions $U(A_{t,a}, a, \Omega_t)$ based on equation (C.1b), using the mapping $\bar{\Omega}$ to compute $\mathrm{E}\left[\bar{\Omega}_{t+1}\right]$.

5. Interpolate the resulting asset prices over the state space grid to construct a new mapping $(r_k(\bar{\omega}), p_l(\bar{\omega}), \tau_c(\bar{\omega}), \pi(\bar{\omega}))$.

6. construct the forecast for the next period state vector from

$$\begin{aligned}
\ln \delta_{t+1} &= \ln \bar{\delta} + \varepsilon_{\delta_{t+1}} \\
\ln \zeta_{L_{t+1}} \mathrm{e}^{-\psi(t+1)} &= \lambda_L \ln \bar{\zeta}_L + (1-\lambda_L) \ln \zeta_{L_t} \mathrm{e}^{-\psi t} + \varepsilon_{L_{t+1}} \\
K_{t+1} &= \mathrm{e}^{-\delta_{t+1}} (K_t + I_t) \\
N_{t+1,a+1} &= (1-\lambda_a) N_{t,a} \\
\bar{N}_{t+1} &= \Gamma_2 \left\{N_{t+1,a}\right\}_{a=-\infty}^{t+1} \\
r_{k_{t+1}} &= r_k(\bar{\Omega}_{t+1}) \\
(1-\lambda_{t-a}) A_{t+1,a} &= \left(1 + r_{k_{t+1}}\right) \left(A_{t,a}^{+} - B_{t,a}\right) + (1 + r_{b_t}) B_{t,a} \\
\bar{A}_{t+1} &= \Gamma \left\{A_{t+1,a}\right\}_{a=-\infty}^{t+1} \\
\bar{\Omega}_{t+1} &= (K_{t+1}, \zeta_{L_{t+1}}, \bar{A}_{t+1}, \bar{N}_{t+1})
\end{aligned}$$

   note that the forecast of $\bar{A}_{t+1}$ involves a simultaneity between r_{k_t} and $\bar{\Omega}_{t+1}$, resulting from the dependency of the saving rate on the asset distribution.

7. go to step 3

D Approximate Aggregation

To restrict the number of state variables that describe the equilibrium, I use an eigenvalue decomposition of the covariance matrix of part of the state vector, viz. the cohort wealth shares $\mathbf{s} = \{s_i\}$, $i \in \{1, \dots n\}$. The adding up constraint implies that $n-1$ wealth shares determine the wealth share vector $\mathbf{s}$. To solve the model, we need to include a 'representative' set of gridpoints from the wealth share vector. Ideally, these gridpoints would be distributed at equidistant probability points in the state space. However, the probability distribution of the wealth shares is not known before the model is solved. The issues addressed in this section are

1. How to construct a distribution of the wealth shares,
2. How to economize on the dimension of the state vector,
3. how to assign the gridpoints symmetrically to the wealth shares, without creating a residual effect (e.g. wrt. the wealth share of the eldest cohort)

Let $\bar{s}_i$ be the benchmark shares (possibly taken from the observed wealth distribution), with $\boldsymbol{\iota}'\bar{\mathbf{s}} = 1$, and take $s_i^* = \bar{s}_i + \varepsilon_i$ as the unconstrained distribution. For arbitrary ε_i, $\mathbf{s}^*$ does not satisfy the adding-up constraint. Furthermore, it may be desirable to impose a correlation structure on ε_i. The problem is to estimate s_i^*, subject to $\boldsymbol{\iota}'\mathbf{s}^* = 1$, using the distributional assumptions on ε_i. The assumption I use is $\varepsilon_{i-1} - 2\varepsilon_i + \varepsilon_{i+1} = u_i \propto N(0, \sigma_i^2)$, $i = 1, \dots, n$. Wealth share profiles are as smooth as the benchmark values, but levels and rates of change over generations may differ.

To discuss the case, define an $(n \times n)$ differentiation matrix $\boldsymbol{\Delta}_2$, with $\boldsymbol{\Delta}_2 (\mathbf{s}^* - \bar{\mathbf{s}}) = \mathbf{u}$, $\mathbf{u} \propto N(0, \boldsymbol{\Sigma})$, and

$$\boldsymbol{\Delta}_2 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

The approximation problem is to find s_i such that $\boldsymbol{\Delta}_2\mathbf{s} \approx \boldsymbol{\Delta}_2\mathbf{s}^*$ in the metric of $\mathbf{u}$, i.e. minimize $(\boldsymbol{\Delta}_2\mathbf{s} - \boldsymbol{\Delta}_2\mathbf{s}^*)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\Delta}_2\mathbf{s} - \boldsymbol{\Delta}_2\mathbf{s}^*)$ wrt. $\mathbf{s}$. Denote $\tilde{\mathbf{s}} = \mathbf{s} - \bar{\mathbf{s}}$. The Lagrangian is

$$\tfrac{1}{2} (\mathbf{u} - \boldsymbol{\Delta}_2\tilde{\mathbf{s}})' \boldsymbol{\Sigma}^{-1} (\mathbf{u} - \boldsymbol{\Delta}_2\tilde{\mathbf{s}}) - \lambda\, \boldsymbol{\iota}'\tilde{\mathbf{s}}$$

and the first-order conditions are

$$\begin{aligned} -\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1} (\mathbf{u} - \boldsymbol{\Delta}_2\tilde{\mathbf{s}}) - \lambda\boldsymbol{\iota} &= 0 \\ \boldsymbol{\iota}'\tilde{\mathbf{s}} &= 0 \end{aligned}$$

It follows that

$$\lambda = \frac{-\boldsymbol{\iota}'\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\mathbf{u}}{\boldsymbol{\iota}'\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\boldsymbol{\iota}}$$

$$\tilde{\mathbf{s}} = \left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\left(\mathbf{I} - \frac{\boldsymbol{\iota}\boldsymbol{\iota}'\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}}{\boldsymbol{\iota}'\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\boldsymbol{\iota}}\right)\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\mathbf{u}$$

The covariance matrix of $\tilde{\mathbf{s}}$ is given by

$$\boldsymbol{\Omega} = \left(\mathbf{I} - \frac{\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\boldsymbol{\iota}\boldsymbol{\iota}'}{\boldsymbol{\iota}'\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}\boldsymbol{\iota}}\right)\left(\boldsymbol{\Delta}_2'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}_2\right)^{-1}$$

The covariance matrix can be decomposed as

$$\boldsymbol{\Omega} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'$$

The eigenvectors $\mathbf{p}_i$ form a spectral decomposition of the age distribution of wealth that can be used to construct a low-dimensional approximation to the distribution of the deviations from the benchmark profile. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a standard normal vector, then

$$\tilde{\mathbf{s}} = \mathbf{P}\boldsymbol{\Lambda}^{1/2}\varepsilon$$

is distributed with mean zero and covariance matrix $\boldsymbol{\Omega}$, as desired. By ordering the eigenvalues and eigenvectors in decreasing size, we may take an approximation of the form

$$\tilde{\mathbf{s}} \approx \mathbf{P}\boldsymbol{\Lambda}^{1/2}\left(\varepsilon_1, \ldots, \varepsilon_m, 0, \ldots, 0\right)'$$

where $m \ll n$. The accuracy of the approximation depends on the speed with which the eigenvalues fall to zero.

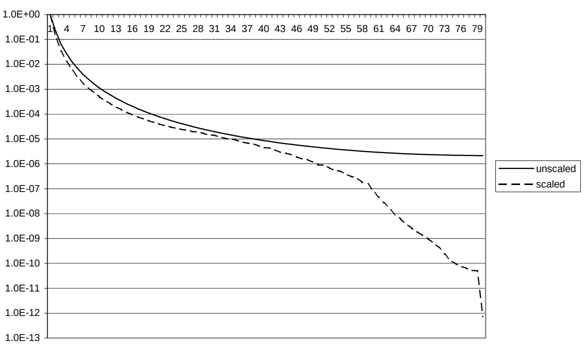

Figure 19: Eigenvalues of the covariance matrix Ω of the asset age profile

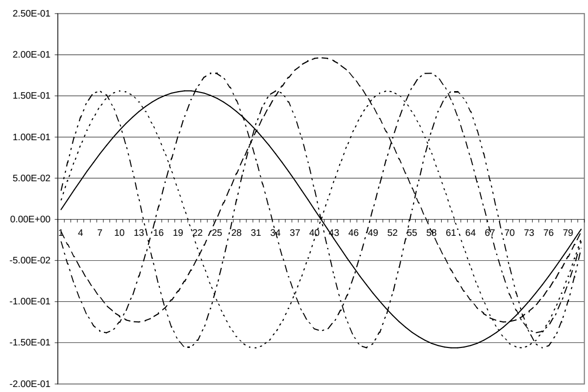

Figure 20: First five eigenvectors of the covariance matrix Ω of the asset age profile

The eigenvalues and eigenvectors are displayed in Figures 19 and 20 for the case of a unit covariance matrix ($\boldsymbol{\Sigma} = \mathbf{I}$). The unscaled eigenvalues fall off to zero quite rapidly, the fifth eigenvalue is only 1/100 of the first one.[23] This suggests that one need use only four of five error terms to approximate the distribution. The approximation can be improved by taking into account the shape of the age-asset profile. Figure 21 shows a "typical" age-asset profile, as it may be generated by the model. The point is that the wealth profile is rather flat over the first twenty years of the working life of a household, as the combined result of a rising wage profile and a precautionary saving motive. This implies that the variation around the benchmark

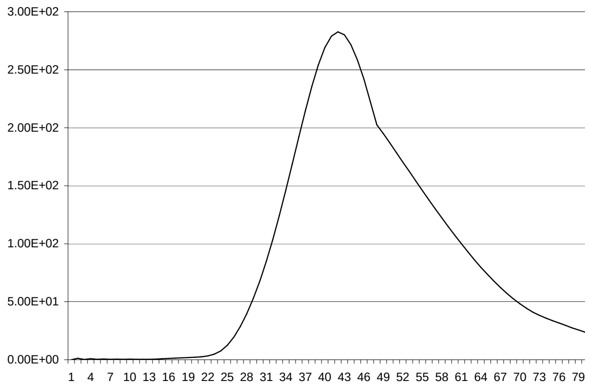

Figure 21: Benchmark age-asset profile

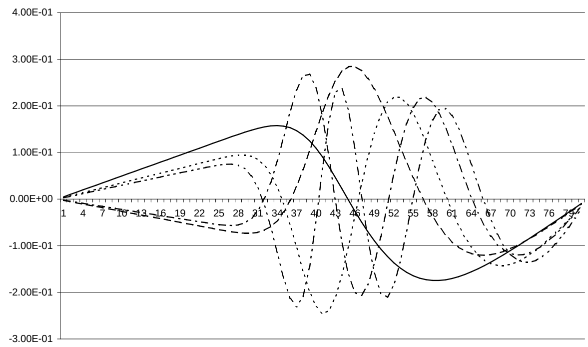

Figure 22: Eigenvectors of the scaled age-asset distribution

profile is lower in the first period of the life of a household, as most of any excess income will be consumed.

This suggests scaling the variances Σ with the asset level of the benchmark profile, i.e. $\sigma_i \propto \bar{s}_i$. With this modification, the eigenvalues fall off faster than without scaling, see Figure 19. The graphs for the eigenvectors changes as given in Figure 22. We see that the effect of the variance scaling is to "stretch" the eigenvectors, so that most of the action occurs for middle-aged households. Theoretically, both young households and, to a lesser extent, old households have few assets. An optimal grid point allocation should take this into account.

[23]The eigenvectors in Figure 20 are in fact the elements of a Fourier sinus expansion of the error series, if extended over the range $(0,\ldots,n+1)$, with $\varepsilon_0 = \varepsilon_{n+1} = 0$, and with the coefficients scaled with the square roots of the eigenvectors.

E Symbol list

A_{t,t_0}	assets of generation t_0
A^+_{t,t_0}	cash-on-hand of generation t_0
B_{t,t_0}	bonds of generation t_0
B_t	total bonds
c_{t,t_0}	consumption of generation t_0
D	depreciation
DIV	dividends
E	firm profits
$h_t(a)$	labor productivity of cohort a in period t
I	investment
K	capital stock
L	employment
l_{t,t_0}	leisure of generation t_0
$l_{\max}$	maximum available time per period
m_{t,t_0}	stochastic discount rate of a household of generation t_0
m^f	stochastic discount rate of the firm
N	population size
n_v	number of shares
p_l	wage rate
p_P	implicit asset price of pension claims
p_v	share price
$q(s_t)$	price of a contingent claim on one consumption unit in state s_t
r_b	return on bonds
r_k	return on equity
s_l	labor share in production in base period
T	government transfers
V	market value of the firm
VN	new share issues
Y	production
y_P	pension

α	$1-\alpha$ is the Arrow-Pratt coefficient of relative risk aversion
γ	intertemporal elasticity of substitution in consumption
$\bar{\delta}$	depreciation rate of capital
δ_P	$\delta_P = 0$ if the household has reached the statutory retirement age, 1 otherwise
ζ_K	capital productivity
ζ_L	labour productivity
θ	elasticity of leisure in full consumption
λ_t	death hazard of a household of age t
λ_I	Lagrange multiplier of the equity short sale constraint
ξ	preference parameter in leisure consumption
ρ	time preference in consumption
$\rho_{\varepsilon_L \varepsilon_\delta}$	correlation between depreciation and productivity disturbances
$\sigma_{\varepsilon_\delta}$	one-period standard deviation of depreciation
σ_{ε_L}	one-period standard deviation of labour productivity
σ_y	substitution elasticity in production
π_P	pension contribution rate
ϕ_P	indexation size of pension claims
φ_i	fertility rate of cohort i
τ_c	consumption tax rate
τ_h	wage income tax rate
ω	paygo replacement rate
Ω	macro state vector

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