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Derivations of DSGE model in CPB memorandum 227

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1 Introduction

This appendix derives the equations for our open economy monopolistic competition model. The model consists of a number of different blocks: households, actuarial insurance firms, foreign investment firms, government, aggregators of goods, and producers. We will discuss each block in turn, beginning with the household block.

2 Households

In our model, we assume that households maximise lifetime utility subject to a budget constraint and a constant probability of death, $1-d$. If the age of a houdeshold in period $t=0$ is denoted a, then its utility function in period t is given by

$$\frac{\left[\left(c_{a+t,t}^{\varsigma} m_{a+t,t}^{1-\varsigma}\right)^{\varphi}\left(1-l_{a+t,t}\right)^{1-\varphi}\right]^{1-\theta}-1}{1-\theta} \tag{2.1}$$

where $c_{a+t,t}$ is the real consumption of the composite good by a household aged $a+t$ in period t, $m_{a+t,t}$ is the real money holdings of the household in period t that was chosen in the previous period, $l_{a+t,t}$ is labour supply, ς is a measure of the utility of holding real money balances, φ is a measure of the disutility of labour, and θ is a measure of the the curvature of the utility function. We have chosen this particular form for the utility function to ensure that we get linear rules for consumption, labour supply and real money balances which can be easily summed across households.

The household budget constraint in nominal terms is

$$\begin{aligned} P_t n_{a+t,t} + P_t m_{a+t,t} = & \frac{\left(1+i_{t-1}\left(1-\tau_{i,t}\right)\right)}{d} P_{t-1} n_{a+t-1,t-1} + \frac{P_{t-1} m_{a+t-1,t-1}}{d} \\ & + \left(1-\tau_{l,t}\right) P_t w_t l_{a+t,t} - \left(1+\tau_{c,t}\right) P_t c_{a+t,t} - P_t \tau_{ls,t} \\ & + \frac{\left(1-\tau_{i,t}\right)}{d} P_t \frac{n_{a+t-1,t-1}}{N_{t-1}} div_t^{AI}, \end{aligned} \tag{2.2}$$

where $P_t m_{a+t,t}$ is the nominal money holdings in period v of a household aged a in period zero. Furthermore, $P_t n_{a+t,t}$ is nominal holdings of actuarial notes and $P_t w_t$ is the nominal wage. The tax on nominal interest income for period t is denoted by $\tau_{i,t}$, on consumption $\tau_{c,t}$, on labour income $\tau_{l,t}$, and the lump-sum tax $\tau_{ls,t}$. The nominal interest rate, i_{t-1}, is defined as the rate agreed in period $t-1$ for deposits made in period $t-1$, that will be paid out in period t. Households also receive a share of the profits made by actuarial insurance firms, $\frac{P_t}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}} div_t^{AI}$, where we assume the share is proportional to actuarial note holdings.

We assume that households place all of their savings with a competitive, zero expected profit actuarial insurance firm. When a household 'dies', its savings becomes the ownership of the actuarial insurance firm. Given the zero expected profit condition imposed by the competitive market, these savings are then in turn redistributed to the living households as part of the return on their investments. Since the return on actuarial notes compensates for the probability of death in this manner, it is always better for the household to let the actuarial insurance firm own other assets on their behalf. This same logic applies to the dividend: only hhouseholds left alive receive a dividend.

We also assume that households have actuarial insurance on their real money holdings. This ensures that the real money holdings of 'deceased' households are also redistributed to the surviving ones. This explains why the first term on the right hand side of (2.2) is divided by d.

Note that in our modelling of households we assume that they do not take into account the riskiness of the assets which the actuarial firm holds on their behalf. That is, they behave as if they have perfect foresight, or in other words, they display certainty equivalence. This means that the model is only valid in a stochastic environment up to a first order approximation, unless we make the certainty equivalence assumption.

Since the household only derives utility from real consumption it is useful to rewrite the budget constraint in current real terms by dividing both sides by P_t:

$$\begin{aligned} n_{a+t,t}+m_{a+t,t} = &\frac{(1+i_{t-1}(1-\tau_{i,t}))}{d}\frac{P_{t-1}}{P_t}n_{a+t-1,t-1}+\frac{P_{t-1}}{P_t}\frac{m_{a+t-1,t-1}}{d}-\tau_{ls,t} \\ &+(1-\tau_{l,t})w_t l_{a+t,t}-(1+\tau_{c,t})c_{a+t,t}+\frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}}div_t^{AI}, \end{aligned}$$

or, letting $\pi_t = \frac{P_t}{P_{t-1}}$ represent inflation,

$$\begin{aligned} n_{a+t,t}+m_{a+t,t} = &\frac{(1+i_{t-1}(1-\tau_{i,t}))}{\pi_t d}n_{a+t-1,t-1}+\frac{m_{a+t-1,t-1}}{\pi_t d}-\tau_{ls,t} \\ &+(1-\tau_{l,t})w_t l_{a+t,t}-(1+\tau_{c,t})c_{a+t,t}+\frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}}div_t^{AI}. \end{aligned} \tag{2.3}$$

Equations (2.1) and (2.3) are theoretically sufficient to completely specify the restricted maximisation problem for households. We would need, however, to consider the first order conditions (FOC) based on the discounted sum of utility over all time periods, with one budget constraint for each time period: no simple task. There is, fortunately, an easier method available. It involves recursively rolling the budget constraints for all time periods into a lifetime budget constraint.

2.1 Lifetime Wealth

The derivation of a lifetime budget constraint involves the creation of a lifetime wealth variable. The trick here, following Ascari and Rankin (2007), is to define the financial wealth variable,

$$fh_{a+t-1,t-1} \equiv \frac{1}{\pi_t d}\left\{[1+i_{t-1}(1-\tau_{i,t})]\,n_{a+t-1,t-1}+m_{a+t-1,t-1}.\right\} \tag{2.4}$$

If we also define net interest income as $ni_t = i_{t-1}(1-\tau_{i,t})$, we then have that

$$\frac{fh_{a+t-1,t-1}\pi_t d}{1+ni_t} = n_{a+t-1,t-1}+m_{a+t-1,t-1}-\frac{ni_t}{1+ni_t}m_{a+t-1,t-1}. \tag{2.5}$$

We can now rewrite the budget constraint as follows.

$$\begin{aligned} n_{a+t,t}+m_{a+t,t} = &fh_{a+t-1,t-1}+(1-\tau_{l,t})w_t l_{a+t,t}-(1+\tau_{c,t})c_{a+t,t} \\ &-\tau_{ls,t}+\frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}}div_t^{AI} \end{aligned} \tag{2.6}$$

Combining equations (2.5) and (2.6) results in the following expression.

$$\frac{fh_{a+t,t}\pi_{t+1}d}{1+ni_{t+1}} = fh_{a+t-1,t-1} + (1-\tau_{l,t})\,w_t l_{a+t,t} - (1+\tau_{c,t})\,c_{a+t,t} - \tau_{ls,t} - \frac{ni_{t+1}}{1+ni_{t+1}} m_{a+t,t} + \frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}} div_t^{AI}$$

How should we interpret this expression? We can rearrange it to make interpretation easier:

$$(1+\tau_{c,t})\,c_{a+t,t} + \frac{ni_{t+1}}{1+ni_{t+1}} m_{a+t,t} + \frac{fh_{a+t,t}\pi_{t+1}d}{1+ni_{t+1}} = fh_{a+t-1,t-1} + (1-\tau_{l,t})\,w_t l_{a+t,t} - \tau_{ls,t} + \frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}} div_t^{AI} \tag{2.7}$$

The right hand side is the real financial wealth brought into the current period plus real wages minus lump-sum taxes, in other words real wealth net-of-taxes. The left hand side is real expenditure net-of-taxes. This is easier to see for the consumption term, but less easy for the other two. Let us return to (2.5), only for the next period, t:

$$\frac{fh_{a+t,t}\pi_{t+1}d}{1+ni_{t+1}} = n_{a+t,t} + m_{a+t,t} - \frac{ni_{t+1}}{1+ni_{t+1}} m_{a+t,t}. \tag{2.8}$$

Now rewrite this expression as

$$n_{a+t,t} + m_{a+t,t} = \frac{fh_{a+t,t}\pi_{t+1}d}{1+ni_{t+1}} + \frac{ni_{t+1}}{1+ni_{t+1}} m_{a+t,t}. \tag{2.9}$$

This demonstrates that the last two terms on the right hand side of (2.7) represent the real cost of the purchasing of actuarial notes and the holding of money balances chosen in period t.

Intuitively, we can think of the third term on the left hand side of (2.7), the financial wealth term, as the real cost of buying $fh_{a+t,t}$ to hold until the next period. This follows from the definition of the real rate of return for the household:

$$\frac{1+ni_{t+1}}{\pi_{t+1}d} = 1 + r_t^h.$$

Substituting this expression into (2.7) makes clear that this looks like the real cost of holding $fh_{a+t,t}$ until the next period. However, this term involves a mistake, because money bears no interest. Therefore, real money balances have an opportunity cost, which is the nominal interest foregone. This cost of holding money balances is

$$ni_{t+1} m_{a+t,t}.$$

The present discounted value of this cost in period t then results in the second term.

We can now combine all the budget constraints into one constraint via substitution. This results in the lifetime budget constraint. Define for convenience

$$y_{a,t} = -\frac{ni_{t+1}}{1+ni_{t+1}} m_{a+t,t} + (1-\tau_{l,t})\,w_t l_{a+t,t} - \tau_{ls,t} + \frac{(1-\tau_{i,t})}{d}\frac{n_{a+t-1,t-1}}{N_{t-1}} div_t^{AI}$$

It follows from earlier definitions that the budget constraint for period t can be rewritten as

$$(1+\tau_{c,t})\,c_{a+t,t}+\frac{\pi_{t+1}d}{1+ni_{t+1}}fh_{a+t,t}=fh_{a+t-1,t-1}+y_{a,t}$$

For period $t+1$, we then obtain

$$fh_{a+t,t}=(1+\tau_{c,t+1})\,c_{a+t,t+1}-y_{a,t+1}+\frac{\pi_t+2d}{1+ni_t+2}fh_{a+t,t+1}$$

Substituting this result into the previous equation for period t results in

$$(1+\tau_{c,t})\,c_{a+t,t}=fh_{a+t-1,t-1}+y_{a,t}-\frac{\pi_{t+1}d}{1+ni_{t+1}}\left[(1+\tau_{c,t+1})\,c_{a+t,t+1}-y_{a,t+1}+\frac{\pi_{t+2}d}{1+ni_{t+2}}fh_{a+t,t+1}\right]$$

For the period $t+2$, the budget constraint can be written as

$$fh_{a+t,t+1}=(1+\tau_{c,t+2})\,c_{a+t,t+2}-y_{a,t+2}+\frac{\pi_{t+3}d}{1+ni_{t+3}}fh_{a+t,t+2}$$

Again substituting this expression into the equation for period t now gives us

$$\begin{aligned}(1+\tau_{c,t})\,c_{a+t,t}&=fh_{a+t-1,t-1}+y_{a,t}-\frac{\pi_{t+1}d}{1+ni_{t+1}}\\ &\times\left[(1+\tau_{c,t+1})\,c_{a+t,t+1}-y_{a,t+1}+\frac{\pi_{t+2}d}{1+ni_{t+2}}\left((1+\tau_{c,t+2})\,c_{a+t,t+2}-y_{a,t+2}+\frac{\pi_{t+3}d}{1+ni_{t+3}}fh_{a+t,t+2}\right)\right]\end{aligned}$$

or

$$\begin{aligned}(1+\tau_{c,t})\,c_{a+t,t}&=fh_{a+t-1,t-1}+y_{a,t}-\frac{\pi_{t+1}d}{1+ni_{t+1}}\left[(1+\tau_{c,t+1})\,c_{a+t,t+1}-y_{a,t+1}\right]\\ &\qquad-\left[\frac{\pi_{t+1}d}{1+ni_{t+1}}\frac{\pi_{t+2}d}{1+ni_{t+2}}\left((1+\tau_{c,t+2})\,c_{a+t,t+2}-y_{a,t+2}+\frac{\pi_{t+3}d}{1+ni_{t+3}}fh_{a+t,t+2}\right)\right]\end{aligned}$$

If we now define

$$\alpha_j^h=\prod_{k=1}^{j}\frac{\pi_k d}{1+ni_k}=\prod_{k=1}^{j}\frac{1}{1+r_{k-1}^h},\qquad \alpha_0=1$$

so that

$$\frac{\alpha_j^h}{\alpha_t^h}=\prod_{k=t}^{j}\frac{\pi_k d}{1+ni_k},\qquad j\geq t, \tag{2.10}$$

then it should be clear that continued substitution results in the following expression for the lifetime budget constraint.

$$\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}(1+\tau_{c,j})\,c_{a+j,j}=fh_{a+t-1,t-1}+\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}y_{a,j}-\lim_{j\to\infty}\frac{\alpha_j^h}{\alpha_t^h}fh_{a+j,j} \tag{2.11}$$

We further impose a No Ponzi Game condition to set the limit term to zero and define household lifetime wealth:

$$\begin{aligned}\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}(1+\tau_{c,j})\,c_{a+j,j}&=fh_{a+t-1,t-1}+\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\left(1-\tau_{l,j}\right)w_j l_j-\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\tau_{ls,j}\\ &\qquad-\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\frac{ni_j}{1+ni_j}m_{a+j,j}+\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\frac{(1-\tau_{i,j})}{d}\frac{n_{a+j-1,j-1}}{N_{j-1}}div_j^{AI}\\ &=h_{a+t,t}\end{aligned} \tag{2.12}$$

Note the term in the lifetime wealth which accounts for the expected cost of holding real money.

2.2 Utility Maximisation

We are now in a position to write down the maximisation problem facing households. The households will want to maximise the discounted sum of utility in all future periods in which they may ‘live’. If the household discount factor is denoted by β, then households will calculate the expected present discounted value at time t of future utility in period $j \geq t$ as

$$(\beta d)^{j-t} \frac{\left[\left(c_{a+j,j}^{\varsigma} m_{a+j,j}^{1-\varsigma}\right)^{\varphi} (1-l_{a+j,j})^{1-\varphi}\right]^{1-\theta} - 1}{1-\theta}. \tag{2.13}$$

Here the factor d^{j-t} represents the probability of the household still being ‘alive’ in period $j \geq t$.

The Lagrangian for the household utility optimisation problem is then given by the following.

$$\begin{aligned} L_t = & \sum_{j=t}^{\infty} (\beta d)^{j-t} \frac{\left[\left(c_{a+j,j}^{\varsigma} m_{a+j,j}^{1-\varsigma}\right)^{\varphi} (1-l_{a+j,j})^{1-\varphi}\right]^{1-\theta} - 1}{1-\theta} \\ & +\lambda \left(h_{a+t,t} - \sum_{k=t}^{\infty} \frac{\alpha_k^h}{\alpha_t^h} (1+\tau_{c,k}) c_{a+k,k} \right) \end{aligned}$$

2.3 Consumption FOC

To avoid unnecessary clutter let us define

$$\Xi_j = \left(c_{a+j,j}^{\varsigma} m_{a+j,j}^{1-\varsigma}\right)^{\varphi} (1-l_{a+j,j})^{1-\varphi}$$

The associated FOC with respect to consumption in period $j \geq t$ is

$$0 = (\beta d)^{j-t} \varsigma\varphi \frac{\Xi_j^{1-\theta}}{c_{a+j,j}} - \lambda \frac{\alpha_j^h}{\alpha_t^h} (1+\tau_{c,j})$$

This leads to the following expression for λ.

$$\lambda = (\beta d)^{j-t} \frac{\varsigma\varphi}{(1+\tau_{c,j})} \frac{\Xi_j^{1-\theta}}{c_{a+j,j}} \frac{\alpha_t^h}{\alpha_j^h} \tag{2.14}$$

2.4 Labour Supply FOC

We can also obtain the FOC with respect to labour supply:

$$0 = -(\beta d)^{j-t} (1-\varphi) \frac{\Xi_j^{1-\theta}}{1-l_{a+j,j}} + \lambda \frac{\alpha_j^h}{\alpha_t^h} \left(1-\tau_{l,j}\right) w_j$$

Rearranging terms leads to the expression

$$1-l_{a+j,j} = (\beta d)^{j-t} \frac{(1-\varphi)}{\lambda} \frac{\alpha_t^h}{\alpha_j^h} \frac{\Xi_j^{1-\theta}}{\left(1-\tau_{l,j}\right) w_j}$$

which, with λ substituted out of the expression, leads to the following equation for labour supply in terms of consumption

$$l_{a+j,j} = 1 - \frac{(1-\varphi)}{\varsigma\varphi} \frac{(1+\tau_{c,j})}{(1-\tau_{l,j})} \frac{c_{a+j,j}}{w_j} \tag{2.15}$$

2.5 Money Demand FOC

The FOC with respect to money is given by

$$0 = (\beta d)^{j-t} (1-\varsigma) \varphi \frac{\Xi_j^{1-\theta}}{m_{a+j,j}} - \lambda \frac{\alpha_j^h}{\alpha_t^h} \frac{ni_{j+1}}{1+ni_{j+1}}$$

This gives us

$$m_{a+j,j} = (\beta d)^{j-t} \Xi_j^{1-\theta} \frac{\varphi(1-\varsigma)}{\lambda} \frac{\alpha_t^h}{\alpha_j^h} \frac{(1+ni_{j+1})}{ni_{j+1}}$$

which simplifies, with λ once again substituted out of the expression, to the following.

$$m_{a+j,j} = (1+\tau_{c,j}) c_{a+j,j} \frac{(1-\varsigma)}{\varsigma} \frac{(1+ni_{j+1})}{ni_{j+1}} \tag{2.16}$$

2.6 Consumption and Wealth

Using the expressions (2.15) and (2.16), we can substitute $l_{a+j,j}$ and $m_{a+j,j}$ out of (2.14) to obtain the following somewhat daunting expression.

$$\frac{\lambda}{(\beta d)^{j-t}} \frac{\alpha_j^h}{\alpha_t^h} \frac{(1+\tau_{c,j})}{\varsigma\varphi} = \frac{\left[\left(c_{a+j,j}^{\varsigma} \left[(1+\tau_{c,j}) \frac{(1-\varsigma)}{\varsigma} \frac{(1+ni_{j+1})}{ni_{j+1}} c_{a+j,j} \right]^{1-\varsigma} \right)^{\varphi} \left(\frac{(1-\varphi)}{\varsigma\varphi} \frac{(1+\tau_{c,j})}{(1-\tau_{l,j})} \frac{c_{a+j,j}}{w_j} \right)^{1-\varphi} \right]^{1-\theta}}{c_{a+j,j}}$$

Combining the terms involving $c_{a+j,j}$ then yields

$$\frac{\lambda}{(\beta d)^{j-t}} \frac{\alpha_j^h}{\alpha_t^h} \frac{(1+\tau_{c,j})}{\varsigma\varphi} = c_{a+j,j}^{-\theta} \left[\left((1+\tau_{c,j}) \frac{(1-\varsigma)}{\varsigma} \frac{(1+ni_{j+1})}{ni_{j+1}} \right)^{\varphi(1-\varsigma)} \left(\frac{(1-\varphi)}{\varsigma\varphi w_j} \frac{(1+\tau_{c,j})}{(1-\tau_{l,i})} \right)^{1-\varphi} \right]^{1-\theta}$$

which can be solved for consumption to give

$$c_{a+j,j} = \lambda^{-\frac{1}{\theta}} (\beta d)^{\frac{j-t}{\theta}} \Omega_j \left(\frac{\alpha_j^h}{\alpha_t^h} \right)^{-\frac{1}{\theta}} \tag{2.17}$$

where

$$\Omega_j =$$
$$\left(\frac{(1+\tau_{c,j})}{\varsigma\varphi}\right)^{-\frac{1}{\theta}}\left[(1+\tau_{c,j})\frac{(1-\varsigma)}{\varsigma}\frac{(1+ni_{j+1})}{ni_{j+1}}\right]^{\delta_1}\left(\left(\frac{1+\tau_{c,j}}{1-\tau_{l,j}}\right)\frac{1-\varphi}{\varsigma\varphi w_j}\right)^{\delta_2},$$

and where

$$\delta_1 = \frac{(1-\varsigma)\,\varphi\,(1-\theta)}{\theta}, \qquad \text{and} \qquad \delta_2 = \tfrac{(1-\varphi)(1-\theta)}{\theta} \tag{2.18}$$

To proceed, substitute (2.17) into the lifetime budget constraint (2.12):

$$\begin{aligned} h_{a+t,t} &= \sum_{k=t}^{\infty}\frac{\alpha_k^h}{\alpha_t^h}(1+\tau_{c,k})\,\lambda^{-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}\,\Omega_k\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{-\frac{1}{\theta}} \\ &= \lambda^{-\frac{1}{\theta}}\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}(1+\tau_{c,k})\,\Omega_k \end{aligned}$$

Rearranging terms, yields

$$\lambda^{\frac{1}{\theta}} = \frac{1}{h_{a+t,t}}\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}(1+\tau_{c,k})\,\Omega_k \tag{2.19}$$

In (2.19), the term $(1+\tau_{c,k})\,\Omega_k$ is equal to the following.

$$(\varsigma\varphi)^{\frac{1}{\theta}}\left(\frac{1-\varsigma}{\varsigma}\right)^{\delta_1}\left(\frac{1-\varphi}{\varsigma\varphi}\right)^{\delta_2}\frac{(1+\tau_{c,k})^{1-\frac{1}{\theta}+\delta_1+\delta_2}}{(w_k\,(1-\tau_{l,k}))^{\delta_2}}\left(\frac{1+ni_{k+1}}{ni_{k+1}}\right)^{\delta_1} \tag{2.20}$$

This expression, however, simplifies further, because

$$1-\frac{1}{\theta}+\delta_1+\delta_2 = -\varsigma\varphi\frac{(1-\theta)}{\theta} \tag{2.21}$$

We have then that

$$(1+\tau_{c,k})\,\Omega_k = \tag{2.22}$$
$$(\varsigma\varphi)^{\frac{1}{\theta}}\left(\frac{1-\varsigma}{\varsigma}\right)^{\delta_1}\left(\frac{1-\varphi}{\varsigma\varphi}\right)^{\delta_2}\left[\frac{(1+\tau_{c,k})^{-\varsigma\varphi}}{(w_k\,(1-\tau_{l,k}))^{(1-\varphi)}}\left(\frac{1+ni_{k+1}}{ni_{k+1}}\right)^{\varphi(1-\varsigma)}\right]^{\frac{(1-\theta)}{\theta}}$$

To make this expression more managable, we introduce the following notation.

$$X^{\star} = (\varsigma\varphi)^{\frac{1}{\theta}}\left(\frac{1-\varsigma}{\varsigma}\right)^{\delta_1}\left(\frac{1-\varphi}{\varsigma\varphi}\right)^{\delta_2} \tag{2.23}$$

$$X_k = \left[\frac{(1+\tau_{c,k})^{-\varsigma\varphi}}{(w_k\,(1-\tau_{l,k}))^{(1-\varphi)}}\left(\frac{1+ni_{k+1}}{ni_{k+1}}\right)^{\varphi(1-\varsigma)}\right]^{\frac{(1-\theta)}{\theta}} \tag{2.24}$$

We can now rewrite (2.19) as follows.

$$\lambda^{-\frac{1}{\theta}} = h_{a+t,t}X^{\star-1}\left[\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}X_k\right]^{-1} \tag{2.25}$$

It turns out that we can define a term that captures the marginal propensity to consume out of wealth:

$$(1-s_j) = X^{\star-1}\left[\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}X_k\right]^{-1}\Omega_j$$

Now rewrite (2.25) in terms of the marginal propensity to consume as

$$\lambda^{-\frac{1}{\theta}} = (1-s_j)\,h_{a+t,t}\Omega_j{}^{-1} \tag{2.26}$$

Returning to consumption, we can now substitute λ out of (2.17), resulting in

$$\begin{aligned} c_{a+j,j} &= (1-s_j)\,h_{a+t,t}\Omega_j{}^{-1}(\beta d)^{\frac{j-t}{\theta}}\Omega_j\left(\frac{\alpha_j^h}{\alpha_t^h}\right)^{-\frac{1}{\theta}} \\ &= (1-s_j)\,h_{a+t,t}\,(\beta d)^{\frac{j-t}{\theta}}\left(\frac{\alpha_j^h}{\alpha_t^h}\right)^{-\frac{1}{\theta}} \end{aligned} \tag{2.27}$$

This demonstrates that consumption is linear in wealth. This expression tells us what the household's optimal plan for all future consumption is starting from period t.

2.7 Aggregation

Having derived household quantities for a given age cohort, we must now aggregate over cohorts to obtain totals.

If we consider consumption at time period t, then (2.27) simplifies to

$$c_{a+t,t} = (1-s_t)\,h_{a+t,t}, \tag{2.28}$$

where $h_{a+t,t}$ is defined in (2.12), and where

$$(1-s_t) = X^{\star-1}\left[\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}X_k\right]^{-1}\Omega_t. \tag{2.29}$$

Note, however, that we can use the expressions (2.22) (2.23) to simplify Ω_t as follows.

$$\begin{aligned} \Omega_t &= (1+\tau_{c,t})\,\Omega_t\,(1+\tau_{c,t})^{-1} = \\ &\quad (\varsigma\varphi)^{\frac{1}{\theta}}\left(\frac{1-\varsigma}{\varsigma}\right)^{\delta_1}\left(\frac{1-\varphi}{\varsigma\varphi}\right)^{\delta_2}\left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi-\theta/(1-\theta)}}{(w_t\,(1-\tau_{l,t}))^{(1-\varphi)}}\left(\frac{1+ni_{t+1}}{ni_{t+1}}\right)^{\varphi(1-\varsigma)}\right]^{\frac{(1-\theta)}{\theta}} \\ &= X^{\star}\left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi-\theta/(1-\theta)}}{(w_t\,(1-\tau_{l,t}))^{(1-\varphi)}}\left(\frac{1+ni_{t+1}}{ni_{t+1}}\right)^{\varphi(1-\varsigma)}\right]^{\frac{(1-\theta)}{\theta}} \end{aligned} \tag{2.30}$$

By combining (2.29) and (2.30), we can express the marginal propensity to consume out of wealth, $1-s_t$, as

$$\left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi-\theta/(1-\theta)}}{(w_t(1-\tau_{l,t}))^{(1-\varphi)}}\left(\frac{1+ni_{t+1}}{ni_{t+1}}\right)^{\varphi(1-\varsigma)}\right]^{\frac{(1-\theta)}{\theta}}\left[\sum_{k=t}^{\infty}\left(\frac{\alpha_k^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{k-t}{\theta}}X_k\right]^{-1} \tag{2.31}$$

We can now see that the marginal propensity to consume is constant over all age cohorts, because it is not a function of age. But the consumption of each cohort, given in (2.28), is simply the product of the marginal propensity to consume multiplied by wealth. If the marginal propensity to consume is constant over all age cohorts, then total consumption, C_t, is given by the product of total wealth, H_t, and the marginal propensity to consume:

$$C_t = (1-s_t)H_t. \tag{2.32}$$

If we take (2.32) as given for the moment, without bothering to worry about the epression for H_t, then we can see that the aggregate labour supply, given (2.15), (2.28), and (2.32), is

$$L_t = \frac{1}{1-d} - \frac{(1-\varphi)}{\varsigma\varphi}\frac{(1+\tau_{c,t})}{(1-\tau_{l,t})}\frac{(1-s_t)}{w_t}H_t,$$

and aggregate money demand, given (2.16), (2.28), and (2.32), is

$$M_t = (1+\tau_{c,t})\frac{(1-\varsigma)}{\varsigma}\frac{1+ni_{t+1}}{ni_{t+1}}(1-s_t)H_t.$$

At this stage it might be sensible to do a quick check - what happens to the demand for real money balances when the nominal interest rate goes up? Well, as ni_{t+1} increases demand for real money balances falls, which is what we want.

Before returning to the derivation of the epression for H_t, we first note that we will assume that there is one household 'born' each period. As a result there will be a total of

$$\sum_{j=0}^{\infty} d^j = \frac{1}{1-d} \tag{2.33}$$

households in equilibrium. This means, for example, that the total lump sum taxes collected by the government in period t will be $\frac{\tau_{ls,j}}{1-d}$.

To obtain an expression for total wealth, H_t, we must return to (2.11) and (2.12). From these two expressions we can see that aggregate wealth can be defined as

$$H_t = HW_t + FH_{t-1},$$

where aggregate human wealth, $HW_t = \sum_a \sum_{j=t}^{\infty} \frac{\alpha_j^h}{\alpha_t^h} y_{a,j}$, is defined to incorporate the cost of money holding:

$$\begin{aligned} HW_t = {} & \sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\left(1-\tau_{l,j}\right)w_j L_j - \frac{1}{1-d}\sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\tau_{ls,j} - \sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}\frac{ni_j}{1+ni_j}M_j \\ & + \sum_{j=t}^{\infty}\frac{\alpha_j^h}{\alpha_t^h}(1-\tau_{i,j})\,div_j^{AI}. \end{aligned}$$

And aggregate financial wealth is defined from individual financial wealth in (2.4) as

$$FH_{t-1} = \frac{1}{\pi_t d}\left[(1+ni_t)N_{t-1} + M_{t-1}\right].$$

2.8 Getting rid of the infinite sums

At this stage we need to get rid of the infinte sums in the equations for human wealth HW_t and for the marginal propensity to consume $(1-s_t)$. Let us start with the easier of the two: human wealth.

$$\begin{aligned}HW_t = {} & \sum_{v=t}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}(1-\tau_{l,v})w_v L_v - \frac{1}{1-d}\sum_{v=t}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}\tau_{ls,v} - \sum_{v=t}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}\frac{i_{v-1}(1-\tau_{i,v})}{1+i_{v-1}(1-\tau_{i,v})}M_v \\ & + \sum_{v=t}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}(1-\tau_{i,v})\,div_v^{AI}\end{aligned}$$

Writing out the first terms of the sums gives:

$$\begin{aligned}HW_t = {} & (1-\tau_{l,t})w_t L_t - \frac{1}{1-d}\tau_{ls,t} - \frac{i_t(1-\tau_{i,t})}{1+i_t(1-\tau_{i,t})}M_t + (1-\tau_{i,t})\,div_t^{AI} \\ & + \sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}(1-\tau_{l,v})w_v L_v - \frac{1}{1-d}\sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}\tau_{ls,v} - \sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}\frac{i_{v-1}(1-\tau_{r,v})}{1+i_{v-1}(1-\tau_{r,v})}M_v \\ & + \sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}(1-\tau_{i,v})\,div_v^{AI}\end{aligned}$$

Adjusting the starting period for the remaining sums and using $\alpha_t^h = \alpha_{t+1}^h\left(1+r_t^h\right)$:

$$\begin{aligned}HW_t = {} & (1-\tau_{l,t})w_t L_t - \tfrac{1}{1-d}\tau_{ls,t} - \tfrac{i_{t-1}(1-\tau_{r,t})}{1+i_{t-1}(1-\tau_{r,t})}M_t + (1-\tau_{i,t})\,div_t^{AI} \\ & + \tfrac{1}{1+r_t^h}\left[\begin{array}{l}\sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_{t+1}^h}(1-\tau_{l,v})w_v L_v - \frac{1}{1-d}\sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_{t+1}^h}\tau_{ls,v} - \sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_{t+1}^h}\frac{i_{v-1}(1-\tau_{i,v})}{1+i_{v-1}(1-\tau_{i,v})}M_v \\ + \sum_{v=t+1}^{\infty}\frac{\alpha_v^h}{\alpha_t^h}(1-\tau_{i,v})\,div_v^{AI}\end{array}\right]\end{aligned}$$

Substituting in the definition of human wealth and real interest for households:

$$\begin{aligned}HW_t = {} & (1-\tau_{l,t})w_t L_t - \frac{1}{1-d}\tau_{ls,t} - \frac{i_{t-1}(1-\tau_{i,t})}{1+i_{t-1}(1-\tau_{i,t})}M_t \\ & + (1-\tau_{i,t})\,div_t^{AI} + \frac{d\pi_{t+1}}{1+i_t(1-\tau_{i,t+1})}HW_{t+1}\end{aligned}$$

Now let's get our hands dirty with the marginal propensity to consume. We denote the bracketed infinite sum on the right hand side of the expression for the marginal propensity to consume, (2.31), as

$$Y_t \equiv \sum_{v=t}^{\infty}\left(\frac{\alpha_v^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{v-t}{\theta}}X_v = X_t + \sum_{v=t+1}^{\infty}\left(\frac{\alpha_v^h}{\alpha_t^h}\right)^{1-\frac{1}{\theta}}(\beta d)^{\frac{v-t}{\theta}}X_v.$$

We can rewrite this as

$$Y_t = X_t + (\beta d)^{\frac{1}{\theta}} \sum_{v=t+1}^{\infty} \left(\frac{\alpha_v^h}{\alpha_{t+1}^h \left(1+r_t^h\right)} \right)^{1-\frac{1}{\theta}} (\beta d)^{\frac{v-(t+1)}{\theta}} X_v,$$

or

$$Y_t = X_t + (\beta d)^{\frac{1}{\theta}} (1+r_t)^{\frac{1}{\theta}-1} Y_{t+1}.$$

Now define the inverse of the first bracketed term on the right hand side of (2.31) as

$$Z_t = \left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi-\theta/(1-\theta)}}{(w_t(1-\tau_{l,t}))^{(1-\varphi)}} \left(\frac{1+ni_{t+1}}{ni_{t+1}} \right)^{\varphi(1-\varsigma)} \right]^{-\frac{(1-\theta)}{\theta}}.$$

This enables us to rewrite (2.31) simply as follows.

$$(1-s_t)^{-1} = Z_t Y$$

$$(1-s_t)^{-1} = Z_t \left(X_t + (\beta d)^{\frac{1}{\theta}} (1+r_t)^{\frac{1}{\theta}-1} Y_{t+1} \right)$$

$$(1-s_t)^{-1} = Z_t X_t + \frac{Z_t}{Z_{t+1}} (\beta d)^{\frac{1}{\theta}} (1+r_t)^{\frac{1}{\theta}-1} Z_{t+1} Y_{t+1}$$

$$(1-s_t)^{-1} = Z_t X_t + \frac{Z_t}{Z_{t+1}} (\beta d)^{\frac{1}{\theta}} (1+r_t)^{\frac{1}{\theta}-1} (1-s_{t+1})^{-1}$$

2.9 Household block

Therefore, the following nine equations describe optimal household behaviour.

Optimal consumption

$$C_t = (1-s_t) H_t \tag{2.34}$$

Total wealth (note how we have changed the time subscript for financial wealth without consequence)

$$H_t = HW_t + FW_t \tag{2.35}$$

Human wealth

$$HW_t = \begin{array}{l} (1-\tau_{l,t}) w_t L_t + (1-\tau_{i,t}) div_t^{AI} - \frac{1}{1-d} \tau_{ls,t} \\ -\frac{i_{t-1}(1-\tau_{i,t})}{1+i_{t-1}(1-\tau_{i,t})} M_t + \frac{d\pi_{t+1}}{1+i_t(1-\tau_{i,t+1})} HW_{t+1} \end{array} \tag{2.36}$$

Financial wealth

$$FW_t = \frac{1}{\pi_t d} \left\{ [1+ni_t] N_{t-1} + M_{t-1} \right\} \tag{2.37}$$

Marginal propensity to consume

$$(1-s_t)^{-1} = Z_t X_t + \frac{Z_t}{Z_{t+1}} (\beta d)^{\frac{1}{\theta}} \left(1+r_t^h\right)^{\frac{1}{\theta}-1} (1-s_{t+1})^{-1} \tag{2.38}$$

where

$$X_t = \left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi}}{(w_t(1-\tau_{l,t}))^{(1-\varphi)}} \left(\frac{1+ni_{t+1}}{ni_{t+1}} \right)^{\varphi(1-\varsigma)} \right]^{\frac{(1-\theta)}{\theta}} \tag{2.39}$$

and

$$Z_t = \left[\frac{(1+\tau_{c,t})^{-\varsigma\varphi-\theta/(1-\theta)}}{(w_t(1-\tau_{l,t}))^{(1-\varphi)}} \left(\frac{1+ni_{t+1}}{ni_{t+1}} \right)^{\varphi(1-\varsigma)} \right]^{-\frac{(1-\theta)}{\theta}} \tag{2.40}$$

Labour supply

$$L_t = \frac{1}{1-d} - \frac{(1-\varphi)}{\varsigma\varphi} \frac{(1+\tau_{c,t})}{(1-\tau_{l,t})} \frac{(1-s_t)}{w_t} H_t \tag{2.41}$$

Money demand

$$M_t = (1+\tau_{c,t}) \frac{(1-\varsigma)}{\varsigma} \frac{1+ni_{t+1}}{ni_{t+1}} (1-s_t) H_t \tag{2.42}$$

3 Actuarial insurance firms

Actuarial insurance firms take household savings and allocate them across competing assets. The assets are government bonds, shares in the foreign investment firm, shares in production firms and direct investment in the capital stock. Production firms rent capital from actuarial firms who own the capital. Production firms pay a rent r_t^k to actuarial firms over the capital decided to rent in the previous period $t-1$. Investments are defined as

$$I_t = K_t - K_{t-1} + \delta K_{t-1}$$

Investments are subject to adjustment costs which are represented by $\psi\left(\frac{I_t}{K_{t-1}} - \delta\right)$. This form enables them to be zero in the steady state. Let the constant number of shares in the representative firm be Z_t, the real share price be q_t and the real dividend paid per share be div_t, then the nominal profits of the actuarial firm for period t are

$$\begin{aligned} P_t\Pi_t^{AI} = {} & P_tN_t - (1+i_{t-1})P_{t-1}N_{t-1} - P_tB_t + \left(1+i_{t-1}^g\right)P_{t-1}B_{t-1} - q_tP_tZ_t + (q_t + div_t)P_tZ_{t-1} \\ & - qf_tP_tZf_t + (qf_t + divf_t)P_tZf_{t-1} + r_t^kP_tK_t - \left[1+\psi\left(\frac{I_t}{K_{t-1}} - \delta\right)\right]P_tI_t \end{aligned}$$

Dividing through by prices again and using the definition of the real interest rate:

$$\begin{aligned} \Pi_t^{AI} = {} & N_t - (1+r_{t-1})N_{t-1} - B_t + \left(1+r_{t-1}^g\right)B_{t-1} - q_tZ_t + (q_t + div_t)Z_{t-1} \\ & - qf_tZf_t + (qf_t + divf_t)Zf_{t-1} + r_t^kK_t - \left[1+\psi\left(\frac{I_t}{K_{t-1}} - \delta\right)\right]I_t \end{aligned}$$

It is worth a moment to think about why this doesn't have inflation terms in whilst most of the stuff we have previously seen did. Basically, whatever capital survives after depreciation can simply be sold at today's prices (remember, capital goods are produced one-for-one from consumption goods), whilst depreciation works on the real capital stock. So that is why we don't see any inflation terms here.

Actuarial firms discount future real profits by expected real return on actuarial notes $1+r_{t-1}$. The reason why we use this discount factor rather than the standard stochastic discount factor in representative agent models is that differently aged households have different levels of consumption, so different levels of riskyness across assets held by the actuarial insurance firm would need to be discounted by a different stochastic discount factor for each household, unless we assume certainty equivalence.

This gives the Langrangian

$$L_v = \sum_{t=v}^{\infty} \alpha_t \left(\begin{array}{l} N_t - (1+r_{t-1}) N_{t-1} - B_t + \left(1+r_{t-1}^g\right) B_{t-1} - q_t Z_t + (q_t + div_t) Z_{t-1} \\ -q f_t Z f_t + (q f_t + div f_t) Z f_{t-1} + r_t^k K_{t-1} - \left[1 + \psi\left(\frac{I_t}{K_{t-1}} - \delta\right)\right] I_t \\ +\Lambda_t \left(I_t + (1-\delta) K_{t-1} - K_t\right) \end{array} \right)$$

where

$$\alpha_t = \frac{1}{\prod_{i=1}^{t} (1+r_{i-1})}$$

The FOC with respect to government debt holding is:

$$1 + r_t = 1 + r_t^g$$

From now on, we will impose this directly. The FOC with respect to production firm share holding is:

$$1 + r_t = \frac{q_{t+1} + div_{t+1}}{q_t}$$

The FOC with respect to foreign investment firm share holding is:

$$1 + r_t = \frac{q f_{t+1} + div f_{t+1}}{q f_t}$$

The FOC with respect to capital:

$$(1+r_t) \Lambda_t = r_{t+1}^k + \left(\frac{I_{t+1}}{K_t}\right)^2 \psi' \left(\frac{I_{t+1}}{K_t} - \delta\right) + \Lambda_{t+1} (1-\delta)$$

A representative actuarial firm is modelled so it considers the received dividend as fixed.

The FOC with respect to investment is:

$$\Lambda_t = \frac{\partial}{\partial I_t} \left\{ \left[1 + \psi\left(\frac{I_t}{K_{t-1}} - \delta\right)\right] I_t \right\}$$

which means

$$\Lambda_t = 1 + \psi\left(\frac{I_t}{K_{t-1}} - \delta\right) + \frac{I_t}{K_{t-1}} \psi' \left(\frac{I_t}{K_{t-1}} - \delta\right)$$

Since the actuarial firms are perfectly competitive, a zero expected profit condition will also hold, which, when we impose $Z_t = Z f_t = 1$ is:

$$\begin{aligned} 0 = {} & N_{t+1} - (1+r_t) N_t - B_{t+1} + (1+r_t) B_t + div_{t+1} + div f_{t+1} \\ & + r_{t+1}^k K_t - \left[1 + \psi\left(\frac{I_{t+1}}{K_t} - \delta\right)\right] I_{t+1} \end{aligned}$$

3.1 Actuarial firms block

The following equations, therefore, describe optimal behaviour on the part of actuarial insurance firms.

Demand for production shares

$$1+r_t = \frac{q_{t+1}+div_{t+1}}{q_t} \tag{3.1}$$

Demand for foreign investment firm shares

$$1+r_t = \frac{q\,f_{t+1}+div\,f_{t+1}}{q\,f_t} \tag{3.2}$$

Optimal investment (1)

$$\Lambda_t = 1+\Psi\left(\frac{I_t}{K_{t-1}}-\delta\right)+\frac{I_t}{K_{t-1}}\Psi'\left(\frac{I_t}{K_{t-1}}-\delta\right) \tag{3.3}$$

Optimal investment (2)

$$\Lambda_t = \frac{1}{1+r_t}\left[r^k_{t+1}+\left(\frac{I_{t+1}}{K_t}\right)^2\Psi'\left(\frac{I_{t+1}}{K_t}-\delta\right)+\Lambda_{t+1}(1-\delta)\right] \tag{3.4}$$

Dividend

$$\begin{aligned} div^{AI}_t = \; & N_t-(1+r_{t-1})N_{t-1}-B_t+(1+r_{t-1})B_{t-1}+div_t+div\,f_t \\ & +r^k_t K_{t-1}-\left[1+\psi\left(\tfrac{I_t}{K_{t-1}}-\delta\right)\right]I_t \end{aligned} \tag{3.5}$$

Zero expected profit condition

$$\begin{aligned} 0 = \; & N_{t+1}-(1+r_t)N_t-B_{t+1}+(1+r_t)B_t+div_{t+1}+div\,f_{t+1} \\ & +r^k_{t+1}K_t-\left[1+\psi\left(\tfrac{I_{t+1}}{K_t}-\delta\right)\right]I_{t+1} \end{aligned} \tag{3.6}$$

Definition of investment

$$I_t = K_t-(1-\delta)K_{t-1} \tag{3.7}$$

Adjustment cost function

$$\Psi\left(\frac{I_t}{K_{t-1}}-\delta\right) = cp\times\left(\frac{I_t}{K_{t-1}}-\delta\right)^2 \tag{3.8}$$

4 Foreign investment firms

Foreign investment firms take funds from the actuarial insurance firms and buy risk-free foreign bonds. Due to the fixed exchange rate between the domestic economy and the rest of the world, we can use the domestic CPI to price foreign bonds. That is, from the point of view of domestic residents, the real value of their foreign bond holdings is the quantity of the domestic good that they can buy. The total dividend they pay their shareholders (which is per period profit) is now:

$$P_t div f_t Z f_{t-1} = \left(1 + i^{fo}_{t-1}\right) P_{t-1} FB_{t-1} - P_t FB_t - P_t \xi \left(\Delta FB_t\right)$$

and the definition of change of real foreign bond holdings is:

$$\Delta FB_t = FB_t - FB_{t-1}$$

So why do we choose this specification for adjustment costs, ie. as a function of the change in real foreign bond holdings? The simple answer is for ease of manipulation, since if we divide through both sides of the dividend by current prices we get the following Lagrangian:

$$F_t = \sum_{v=t}^{\infty} \frac{\alpha_v}{\alpha_t} \left[r^{fo}_{v-1} FB_{v-1} - \Delta FB_v - \xi \left(\Delta FB_v\right) + \Lambda f_v \left(\Delta FB_v - FB_v + FB_{v-1}\right) \right]$$

The variables α_v and α_t are still defined as the compound real interest rate obtained from the domestic inflation and the domestic nominal interest rate: The FOC with respect to ΔFB_t is:

$$\Lambda f_t = 1 + \left\{ \frac{\partial}{\partial \Delta FB_t} \left[\xi \left(\Delta FB_t\right)\right] \right\} = 1 + \xi' \left(\Delta FB_t\right)$$

The FOC with respect to FB_t:

$$\Lambda f_t = \frac{1}{1 + r_t} \left[r^{fo}_t + \Lambda f_{t+1} \right]$$

We can also define the capital account:

$$NX_t = FB_t - \left(1 + r^{fo}_{t-1}\right) FB_{t-1}$$

4.1 Foreign investment firms block

Optimal behaviour of foreign investment firms is therefore given by the following equations.

Optimal foreign bond holding (1)

$$\Lambda f_t = 1 + \xi' \left(\Delta FB_t\right) \tag{4.1}$$

Optimal foreign bond holding (2)

$$\Lambda f_t = \frac{1}{1 + r_t} \left(r^{fo}_t + \Lambda f_{t+1} \right) \tag{4.2}$$

Definition of real dividend

$$divf_t = r^{fo}_{t-1} FB_{t-1} - \Delta FB_t - \xi\left(\Delta FB_t\right) \tag{4.3}$$

Definition of foreign bond adjustment

$$\Delta FB_t = FB_t - FB_{t-1} \tag{4.4}$$

Adjustment cost function

$$\xi\left(\Delta FB_t\right) = cpf \times \left(\Delta FB_t\right)^2 \tag{4.5}$$

Real net exports definition: Capital account

$$NX_t = FB_t - \left(1 + r^{fo}_{t-1}\right) FB_{t-1} \tag{4.6}$$

5 Government

Let us first start of by working with the government budget constraint covering both fiscal and monetary authoritles. Whilst the Netherlands does not have it's own monetary policy, it still enjoys transfers of seignorage revenues from the euro system, hence we need to model this in our analysis. One must also note that, if households do not value money, which is achieved by setting $\varsigma = 1$ in the utility function, then there will not be any seignorage revenues since there will not be any money. Central banks can inject money into the economy by performing open market operations. That is, they use newly printed money to buy nominal bonds. Typically, however, monetary economics models changes in the money supply as cash-injections direct to households. That is, every household wakes up each period with their share of the increase in the money supply posted through their letter box. This is because, in models with infinitely-lived representative agents and non-distorting taxation, Ricardian equivalence holds and bond holdings are irrelevant. Hence, cash-in-the-letterbox and open market operations are equivalent. We have both finitely-lived agents and distortionary taxation, so we will need to model open market operations. Let us define $P_t B_t$ as the nominal price of a bond in period t, then the budget constraint of the monetary authority looks like:

$$P_t M_t - P_{t-1} M_{t-1} + i_{t-1} P_{t-1} B^M_{t-1} = P_t B^M_t - P_{t-1} B^M_{t-1} + P_t Transfer_t$$

Here $P_t B^M_t$ are the nominal bond holdings of the monetary authority. The LHS is the nominal income of the monetary authority: the nominal money issued plus the nominal interest received on bond holdings. The RHS are the outgoings: the increase in nominal bond holdings plus the transfers to the fiscal authority.

If we now turn to the budget constraint of the fiscal authority. For ease of notation, let us define a term to represent all nominal tax receipts, $P_t T_t$:

$$P_t T_t = \tau_{i,t} i_{t-1} P_{t-1} N_{t-1} + \tau_{l,t} P_t w_t L_t + \tau_{c,t} P_t C_t + \frac{1}{1-d} P_t \tau_{ls,t} + \tau_{i,t} P_t div^{AI}_t$$

the fiscal authority budget constraint is then:

$$P_t G_t + i_{t-1} P_{t-1} B^T_{t-1} = P_t B^T_t - P_{t-1} B^T_{t-1} + P_t T_t + P_t Transfer_t$$

Here $P_t B^T_t$ are the total of all outstanding nominal bonds. Using the definition:

$$B^M_t + B_t = B^T_t$$

where $P_t B_t$ are the nominal bond holdings of the public, we can combine the two budget constraints to get a single budget constraint for the government as a whole. First we use the definition of total bonds:

$$P_t G_t + i_{t-1} P_{t-1} B_{t-1} + i_t P_{t-1} B^M_{t-1} = P_t B_t - P_{t-1} B_{t-1} + P_t B^M_t - P_{t-1} B^M_{t-1} + P_t T_t + P_t Transfer_t$$

then substitute out the transfers from the monetary authority to the fiscal authority:

$$P_t M_t - P_{t-1} M_{t-1} + i_{t-1} P_{t-1} B^M_{t-1} - \left(P_t B^M_t - P_{t-1} B^M_{t-1}\right) = P_t Transfer_t$$

so

$$\begin{aligned} P_t G_t + i_{t-1} P_{t-1} B_{t-1} + i_{t-1} P_{t-1} B^M_{t-1} = {} & P_t B_t - P_{t-1} B_{t-1} + P_t B^M_t - P_{t-1} B^M_{t-1} \\ & + P_t T_t + P_t M_t - P_{t-1} M_{t-1} + i_{t-1} P_{t-1} B^M_{t-1} \\ & - \left(P_t B^M_t - P_{t-1} B^M_{t-1}\right) \end{aligned}$$

Cancelling terms leaves us with the consolidated government budget constraint:

$$P_t G_t + i_{t-1} P_{t-1} B_{t-1} = P_t B_t - P_{t-1} B_{t-1} + P_t T_t + P_t M_t - P_{t-1} M_{t-1}$$

If we look at this we can see that there is interaction between fiscal and monetary policy because the seignorage revenues enter into the budget constraint. Again, it is often useful to write this in real terms by dividing through by P_t.

$$G_t + i_{t-1} \frac{B_{t-1}}{\pi_t} = B_t - \frac{B_{t-1}}{\pi_t} + T_t + M_t - \frac{M_{t-1}}{\pi_t}$$

Or rearranging and using the relevant real interest rate definitions from previously:

$$B_t = G_t + (1 + r_{t-1}) B_{t-1} - \tau_{i,t} \frac{i_{t-1}}{\pi_t} N_{t-1} - \tau_{l,t} w_t L_t - \tau_{c,t} C_t - \frac{1}{1-d} \tau_{ls,t} + \tau_{i,t} div^{AI}_t - M_t + \frac{M_{t-1}}{\pi_t}$$

We also need to specify a fiscal policy rule so that debt remains bounded. The following example uses lump-sum taxes to pay off slightly more than the interest burden on outstanding debt:

$$\frac{1}{1-d} \tau_{ls,t} = (1 + r_{t-1} + \tau_{sus}) B_{t-1}$$

Alternatively we can think of many different rules such as the following that adjusts consumption taxes smoothly to target a specific deficit-to-GDP ratio:

$$\tau_{c,t} C_t = \rho_{ts} \tau_{c,t-1} C_{t-1} + \Omega \left[\frac{G_t + (1 + r_{t-1}) B_{t-1} - T_t - \psi_1 Y_t}{Y_t} \right]$$

So basically, the consolidated government has an extra instrument: the growth rate of money. However, the Netherlands doesn't have its own monetary policy so we need to specify a monetary policy rule for our model that takes the institutional set-up into account. Namely, we can set the domestic nominal rate equal to the foreign nominal rate:

$$i_t = i_t^{fo}$$

In fact, we impose this directly through the various interest rate definitions we have introduced..

5.1 Government block

A balanced budget government block is given by the following equations.

Government spending

$$G_t = G_0 + e_t^g \tag{5.1}$$

Government budget constraint

$$B_t = G_t + (1 + r_{t-1}) B_{t-1} - \tau_{i,t} \frac{i_{t-1}}{\pi_t} N_{t-1} - \tau_{l,t} w_t L_t - \tau_{c,t} C_t - \frac{1}{1-d} \tau_{ls,t} + \tau_{i,t} P_t div_t^{AI} \tag{5.2}$$

Fiscal policy rule

$$\frac{1}{1-d} \tau_{ls,t} = (1 + r_{t-1} + \tau_{sus}) B_{t-1} \tag{5.3}$$

6 Aggregators

6.1 Composite domestic and foreign bundles

Aggregate consumption, government expenditure, investment and the associated real costs must be made from the same composite good, Υ_t. The composite good is made up of foreign and domestically produced goods, $\Upsilon_{F,t}$ and $\Upsilon_{H,t}$ respectively, from the following CES aggregator:

$$\Upsilon_t \equiv \left[(1-(1-n)\,\alpha)^{\eta}\,\Upsilon_{H,t}^{1-\eta} + ((1-n)\,\alpha)^{\eta}\,\Upsilon_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

Here, $\eta = 0$ gives perfect substitutability between domestic and foreign goods, α measures the degree of home bias with $\alpha = 1$ indicating no home bias and n gives the size of the domestic economy relative to the rest of the world. Given that the domestically produced good has price P_H and the foreign good price P_F, cost minimisation gives us the price of the composite good as well as the demand for each of the two components.

$$P_t = \left[(1-(1-n)\,\alpha)\,P_{H,t}^{\frac{\eta-1}{\eta}} + (1-n)\,\alpha P_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$\Upsilon_{H,t} = (1-(1-n)\alpha)\left(\frac{P_t}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t$$

$$\Upsilon_{F,t} = (1-n)\alpha\left(\frac{P_t}{P_{F,t}}\right)^{\frac{1}{\eta}}\Upsilon_t$$

Assuming symmetric home bias at home and abroad, we get the following price and demands in the rest of the world

$$P_t^* = \left[n\alpha P_{H,t}^{\frac{\eta-1}{\eta}} + (1-n\alpha)\,P_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$\Upsilon_{H,t}^* = n\alpha\left(\frac{P_t^*}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t^*$$

$$\Upsilon_{F,t}^* = (1-n\alpha)\left(\frac{P_t^*}{P_{F,t}}\right)^{\frac{1}{\eta}}\Upsilon_t^*$$

Note that due to home bias the composite goods will contain different proportions of the two underlying goods and will therefore not necessarily have the same price. The price level can be non-stationary, so we define everything in terms of relative prices:

$$S_t = \frac{P_{F,t}}{P_{H,t}}$$

$$g(S_t) = \left[(1-(1-n)\alpha) + (1-n)\alpha S_t^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} = \frac{P_t}{P_{H,t}}$$

$$g^*(S_t) = \left[(1-n\alpha) + n\alpha S_t^{\frac{1-\eta}{\eta}}\right]^{\frac{\eta}{\eta-1}} = \frac{P_t^*}{P_{F,t}}$$

The terms of trade is still defined in terms of the indivdual levels but we can redefine it as a difference equation:

$$S_t = \frac{P_{F,t}}{P_{H,t}} \frac{P_{H,t-1}}{P_{F,t-1}} \frac{P_{F,t-1}}{P_{H,t-1}}$$

or

$$\frac{S_t}{S_{t-1}} = \frac{\pi_{F,t}}{\pi_{H,t}}$$

We can use the above to derive an expression for the current account:

$$P_t NX_t = P_{H,t}\Upsilon^*_{H,t} - P_{F,t}\Upsilon_{F,t}$$

Or in real terms

$$NX_t = \frac{1}{g(S_t)} n\alpha \left(g^*(S_t) S_t\right)^{\frac{1}{\eta}} \Upsilon^*_t - \frac{S_t}{g(S_t)}(1-n)\alpha \left(\frac{g(S_t)}{S_t}\right)^{\frac{1}{\eta}} \Upsilon_t$$

or

$$NX_t = \frac{1}{g(S_t)} n\alpha \left(g^*(S_t) S_t\right)^{\frac{1}{\eta}} \Upsilon^*_t - (1-n)\alpha \left(\frac{g(S_t)}{S_t}\right)^{\frac{1-\eta}{\eta}} \Upsilon_t$$

We can also use the above definitions to derive the following expression for domestic CPI definition:

$$\pi_t = \frac{g(S_t)}{g(S_{t-1})}\pi_{H,t}$$

6.2 Within domestic bundles

The domestic good that makes up the domestic share in the aggregate composite good is itself a composite of a continuum of domestically produced goods. The consumption bundle $\Upsilon_{H,t}$ represents the domestic consumption of the continuum of domestic varieties on the interval $[0,n]$. The bundle $\Upsilon^*_{H,t}$ represents the foreign consumption of the continuum of domestic varieties on the interval $[0,n]$. Similarly, the consumption bundle $\Upsilon_{F,t}$ equals domestic consumption of the continuum of the foreign varieties which are on the interval $[n,1]$. The foreign consumption $\Upsilon^*_{F,t}$ of such varieties is defined analogously. The bundles denote aggregate quantities. Stated formally we define:

$$\Upsilon_{H,t} \equiv \left[\int_0^n \left(\frac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$$

$$\Upsilon_{F,t} \equiv \left[\int_n^1 \left(\frac{1}{1-n}\right)^{\varepsilon} \Upsilon_{F,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$$

$$\Upsilon^*_{H,t} \equiv \left[\int_0^n \left(\frac{1}{n}\right)^{\varepsilon} \Upsilon^*_{H,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$$

$$\Upsilon^*_{F,t} \equiv \left[\int_n^1 \left(\frac{1}{1-n}\right)^{\varepsilon} \Upsilon^*_{F,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$$

where i represents a particular variety and $0 \leq \varepsilon \leq 1$ represents the inverse of substitution elasticity of the domestic varieties as well as the foreign varieties. An expression for the optimal choice of $\Upsilon_{H,t}(i)$ will be obtained in this section. Expressions for the other three types of variety consumption follow similarly.

The perfectly competitive aggregator takes the prices of varieties as given. They choose the sequence for consumption of varieties $\{\Upsilon_{H,t}(i)\}_{i=0}^{1}$ to minimise the nominal production cost for a certain number of consumption bundles:

$$\min_{\{\Upsilon_{H,t}(i)\}_{i=0}^{1}} \int_0^n P_{H,t}(i)\Upsilon_{H,t}(i) di$$

$$\text{s.t.} \Upsilon_{H,t} \geq \overline{\Upsilon}$$

The corresponding Lagrangian is given by

$$L = \int_0^n P_{H,t}(i)\Upsilon_{H,t}(i) di - P_{H,t}\left(\Upsilon_{H,t} - \overline{\Upsilon}\right)$$

where $P_{H,t}$ denotes the marginal nominal cost of a consumption bundle Υ_H. Perfect competition among aggregators drives the price down to the marginal cost. Differentiating L

with respect to any $\Upsilon_{H,t}(j)$ ($j \in [0,n]$) should be zero. It is straightforward to see that considering a single j provides a zero value for the derivative because each variety producer has measure zero. However, by considering all j in an arbitrary subset Ω (having nonzero measure) on $[0,n]$ we should still find

$$\int\limits_{\Omega} \frac{\partial L}{\partial \Upsilon_{H,t}(j)} dj = \int\limits_{\Omega} P_{H,t}(j) dj - P_{H,t} \int\limits_{\Omega} \frac{\partial \Upsilon_{H,t}}{\partial \Upsilon_{H,t}(j)} dj = 0$$

Note that

$$\int\limits_{\Omega} \frac{\partial \Upsilon_{H,t}}{\partial \Upsilon_{H,t}(j)} dj = \int\limits_{\Omega} \frac{\partial}{\partial \Upsilon_{H,t}(j)} \left[\left(\int\limits_0^n \left(\tfrac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \right] dj$$
$$= \int\limits_{\Omega} \frac{1}{1-\varepsilon} \left(\int\limits_0^n \left(\tfrac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}-1} \left(\int\limits_0^n \frac{\partial}{\partial \Upsilon_{H,t}(j)} \left[\left(\tfrac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}(i)^{1-\varepsilon} \right] di \right) dj$$

$$= \frac{1}{1-\varepsilon} \left(\int\limits_0^n \left(\tfrac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}(i)^{1-\varepsilon} di \right)^{\frac{\varepsilon}{1-\varepsilon}} \int\limits_{\Omega} \left(\tfrac{1}{n}\right)^{\varepsilon} (1-\varepsilon) \Upsilon_{H,t}(j)^{-\varepsilon} dj$$
$$= \left(\tfrac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}^{\varepsilon} \int\limits_{\Omega} \Upsilon_{H,t}(j)^{-\varepsilon} dj$$

Substituting the latter result

$$\int\limits_{\Omega} P_{H,t}(j) dj = P_{H,t} \left(\frac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}^{\varepsilon} \int\limits_{\Omega} \Upsilon_{H,t}(j)^{-\varepsilon} dj$$

Since subset Ω is arbitrary, this relation must hold for any $j \in [0,n]$ resulting in:

$$P_{H,t}(j) = P_{H,t} \left(\frac{1}{n}\right)^{\varepsilon} \Upsilon_{H,t}^{\varepsilon} \Upsilon_{H,t}(j)^{-\varepsilon}$$

This implies the cost minimising choice of domestic variety $j \in [0,n]$ for the domestic economy is:

$$\Upsilon_{H,t}(j) = \frac{1}{n} \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\frac{1}{\varepsilon}} \Upsilon_{H,t}$$

A similar expression can be derived for the foreign economy's demand for a given domestic variety j:

$$\Upsilon_{H,t}^{*}(j) = \frac{1}{n} \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\frac{1}{\varepsilon}} \Upsilon_{H,t}^{*}$$

These expressions are the demand curves faced by each monopolistically competitive domestic production firm j.

6.3 Aggregators block

The aggregators give us the following equations for the final model:

Evolution of terms of trade

$$\frac{S_t}{S_{t-1}} = \frac{\pi_{F,t}}{\pi_{H,t}} \tag{6.1}$$

The domestic CPI to PPI ratio

$$g(S_t) = \left[(1 - (1-n)\,\alpha) + (1-n)\,\alpha S_t^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{6.2}$$

The foreign CPI to PPI ratio

$$g^*(S_t) = \left[(1 - n\alpha) + n\alpha S_t^{\frac{1-\eta}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{6.3}$$

Real net exports definition: Current account

$$NX_t = \frac{1}{g(S_t)} n\alpha \left(g^*(S_t)\, S_t \right)^{\frac{1}{\eta}} \Upsilon_t^* - (1-n)\alpha \left(\frac{g(S_t)}{S_t} \right)^{\frac{1-\eta}{\eta}} \Upsilon_t \tag{6.4}$$

Domestic CPI definition

$$\pi_t = \frac{g(S_t)}{g(S_{t-1})} \pi_{H,t} \tag{6.5}$$

7 Production firms

The demand for each intermediate good is given by summing the demand from domestic aggregators and from exports. We can always rewrite the firm's optimisation problem in terms of maximising the difference between marginal revenue and marginal cost, which we will do here without deriving an expression for marginal cost just yet. The production function of the firm is given by:

$$Y_t = A_t K_{t-1}^{\chi} L_t^{1-\chi}$$

Faia and Monacelli (2008) use Rotemberg (1982) sticky prices. That is, changing prices has a real cost given by:

$$\frac{\vartheta}{2}\left(\frac{\bar{P}_{H,t}}{\bar{P}_{H,t-1}}-1\right)^2$$

That is, if the firm wants to change its price it must go and buy some of the domestic composite to cover the costs. So the problem for the firm is to choose a price that will maximise expected profits using a discount factor from their owner. In the Faia and Monacelli paper the household owns the firm and there are complete contingent markets, so the discount factor is the price in period zero of a certain claim on one unit of domestic currency in period t. Whereas for us, the actuarial insurance firms own the firms. We will still call this $Q_{t,t+k}$ since we can put in whatever discount rate we like at a later date. This gives us an expression for the expected nominal profit:

$$E_t \sum_{k=0}^{\infty} Q_{t,t+k}\left[Y_{t+k}(j)\left(\bar{P}_{H,t+k} - MC_{t+k}^n\right) - \frac{\vartheta}{2}\left(\frac{\bar{P}_{H,t+k}}{P_{H,t+k-1}}-1\right)^2 P_{H,t+k}\right]$$

The firm chooses a price to maximise this subject to the demand curve it faces. This is made up by summing the demand from the domestic aggregators and export demand where domestic demand for domestically produced goods is given by

$$\Upsilon_{H,t} = (1-(1-n)\alpha)\left(\frac{P_t}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t$$

and foreign demand for domestically produced goods is given by

$$\Upsilon_{H,t}^* = n\alpha\left(\frac{P_t^*}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t^*$$

where Υ_{t+k} is total domestic demand for the composite good

$$\Upsilon_{t+k} = C_{t+k} + I_{t+k} + G_{t+k} + \Psi\left(\frac{I_t}{K_{t-1}} - \delta\right) I_t + \xi\left(\Delta FB_t\right)$$

Since the firm is small they can take total consumption of all domestically produced goods as given when solving their pricing problem. We can also adjust the definition introduced in the aggregators section to simplify our analysis. We adjust the definition to take into account aggregate price adjustment costs, which are exogenous to the firm under consideration:

$$\Upsilon_{D,t+k} = \Upsilon_{H,t+k} + \Upsilon^*_{H,t+k} + \frac{\vartheta}{2}\left(\pi_{H,t} - 1\right)^2$$

so setting demand equal to supply gives

$$Y_{t+k}\left(j\right) = \Upsilon_{D,t+k}\left(j\right) = \frac{1}{n}\left(\frac{\bar{P}_{H,t+k}}{P_{H,t+k}}\right)^{-\frac{1}{\varepsilon}}\Upsilon_{D,t+k}$$

Rather than mess about with Lagrangians we will impose the constraint directly by substituting the expression we have just derived for total production of the firm under question. The problem of the firm is to choose a price to maximise:

$$E_t \sum_{k=0}^{\infty} Q_{t,t+k}\left[\frac{1}{n}\Upsilon_{D,t+k}\left(\frac{\bar{P}_{H,t+k}}{P_{H,t+k}}\right)^{-\frac{1}{\varepsilon}}\left(\bar{P}_{H,t+k} - MC^n_{t+k}\right) - \frac{\vartheta}{2}\left(\frac{\bar{P}_{H,t+k}}{P_{H,t+k-1}} - 1\right)^2 P_{H,t+k}\right]$$

What is important to note here is that the firm can only change $\bar{P}_{H,t+k}$. The FOC for the optimal price $\bar{P}_{H,t}$ is:

$$\begin{aligned} 0 = {} & Q_{t,t}\tfrac{1}{n}\Upsilon_{D,t}\left(\tfrac{\bar{P}_{H,t}}{P_{H,t}}\right)^{-\frac{1}{\varepsilon}} - Q_{t,t}\tfrac{1}{\varepsilon}\tfrac{1}{n}\Upsilon_{D,t}\left(\tfrac{\bar{P}_{H,t}}{P_{H,t}}\right)^{-\frac{1}{\varepsilon}-1}\tfrac{1}{P_{H,t}}\left(\bar{P}_{H,t} - MC^n_t\right) \\ & - Q_{t,t}\vartheta P_{H,t}\left(\tfrac{\bar{P}_{H,t}}{P_{H,t-1}} - 1\right)\tfrac{1}{\bar{P}_{H,t-1}} + Q_{t,t+1}\vartheta P_{H,t+1}\left(\tfrac{\bar{P}_{H,t+1}}{P_{H,t}} - 1\right)\tfrac{\bar{P}_{H,t+1}}{P^2_{H,t}} \end{aligned}$$

Simplifying:

$$\begin{aligned} 0 = {} & \left(1 - \tfrac{1}{\varepsilon}\right)Q_{t,t}\tfrac{1}{n}\Upsilon_{D,t}\left(\tfrac{\bar{P}_{H,t}}{P_{H,t}}\right)^{-\frac{1}{\varepsilon}} + Q_{t,t}\tfrac{1}{\varepsilon}\tfrac{1}{n}\Upsilon_{D,t}\left(\tfrac{\bar{P}_{H,t}}{P_{H,t}}\right)^{-\frac{1}{\varepsilon}-1}\tfrac{MC^n_t}{P_{H,t}} \\ & - Q_{t,t}\vartheta P_{H,t}\left(\tfrac{\bar{P}_{H,t}}{\bar{P}_{H,t-1}} - 1\right)\tfrac{1}{\bar{P}_{H,t-1}} + Q_{t,t+1}\vartheta P_{H,t+1}\left(\tfrac{\bar{P}_{H,t+1}}{P_{H,t}} - 1\right)\tfrac{\bar{P}_{H,t+1}}{P^2_{H,t}} \end{aligned}$$

Imposing a symmetric equilibrium, $\bar{P}_{H,t} = P_{H,t}$ implies that we can write the optimal price for all firms as:

$$Q_{t,t}\vartheta\pi_{H,t}\left(\pi_{H,t} - 1\right) = \left(1 - \frac{1}{\varepsilon}\right)Q_{t,t}\frac{1}{n}\Upsilon_{D,t} + Q_{t,t}\frac{1}{\varepsilon}\frac{1}{n}\Upsilon_{D,t}\frac{MC^n_t}{P_{H,t}} + Q_{t,t+1}\vartheta\pi^2_{H,t+1}\left(\pi_{H,t+1} - 1\right)$$

Just for now, we want to think about what $Q_{t,t+k}$ is. This is the stochastic discount factor for discounting nominal profits (look at the objective function, it is for nominal profits). In Faia and Monacelli the household is the owner of the firm and the discount factor depends on the ratio of marginal utilities, ie. the price of certain unit of consumption in a given period. However, for us, the firm is owned by the actuarial firm, which raises capital from households by paying the gross nominal interest rate. If you were to compare the two different discount factors you should find that they are the same. In our case, then, $Q_{t,t} = 1$ and $Q_{t,t+1}$ is the gross nominal interest rate from t to $t+1$. Substituting this into our optimal pricing equation and also going to real marginal cost:

$$\vartheta \pi_{H,t}\left(\pi_{H,t}-1\right)=\left(1-\frac{1}{\varepsilon}\right)\frac{1}{n}\Upsilon_{D,t}+\frac{1}{\varepsilon}\frac{1}{n}\Upsilon_{D,t}MC_t^r+\frac{1}{1+i_t}\vartheta \pi_{H,t+1}^2\left(\pi_{H,t+1}-1\right)$$

Faia and Monacelli go on to substitute out output using the production function and marginal cost, which we are not going to do. Comparing their equation 49 with the above equation the main difference is that their stochastic discount factor takes into account expected terms of trade movements, ours doesn't explicitly yet. To see where this comes from consider the following expression:

$$\frac{1}{1+i_t}\pi_{H,t+1}=\frac{1}{1+r_t}\frac{\pi_{H,t+1}}{\pi_{t+1}}$$

Now we will use the expression we derived for the domestically produced goods price level to CPI ratio so:

$$\frac{g\left(S_t\right)}{g\left(S_{t+1}\right)}=\frac{P_t}{P_{t+1}}\frac{P_{H,t+1}}{P_{H,t}}=\frac{\pi_{H,t+1}}{\pi_{t+1}}$$

so we can write our optimal price expression as:

$$\varepsilon\vartheta \pi_{H,t}\left(\pi_{H,t}-1\right)=\frac{1}{n}\Upsilon_{D,t}\left(MC_t^r-1+\varepsilon\right)+\frac{\varepsilon\vartheta}{1+r_t}\frac{g\left(S_t\right)}{g\left(S_{t+1}\right)}\pi_{H,t+1}\left(\pi_{H,t+1}-1\right)$$

This is an exact non-linear New Keynesian Phillips Curve. If you log-linearise this we get the same functional form as the standard NKPC. One point to note is that past inflation does not enter this expression, which was a key ingredient in matching inflation persistence according to Christiano et al. (2005). The pricing equation is already assuming optimal behaviour in factor markets. So what is that behaviour? Remember the firm has set the price, not the quantity. So whatever is demanded at the current price the firm must supply. Whereas in the perfect competition with flexible prices case the firm was choosing quantities, now the firm has set the price and there will be a unique quantity of capital and labour that will minimise the cost of matching demand (and maximise profits). This is given by the cost minimisation problem:

$$\min_{K,L}\left\{g\left(S_t\right)w_tL_t+g\left(S_t\right)r_t^kK_{t-1}\right\}$$

subject to

$$\Upsilon_t=A_tK_{t-1}^{\chi}L_t^{1-\chi}\geq\overline{\Upsilon}$$

Setting up the Lagrangian where the Lagrange multiplier is equal to real marginal cost:

$$g\left(S_t\right)w_tL_t+g\left(S_t\right)r_t^kK_{t-1}-MC_t^r\left(A_tK_{t-1}^{\chi}L_t^{1-\chi}-\overline{\Upsilon}\right)$$

FOC capital demand:

$$g(S_t)\, r_t^k = MC_t^r \chi A_t \left(\frac{K_{t-1}}{L_t}\right)^{\chi-1}$$

FOC labour demand:

$$g(S_t)\, w_t = MC_t^r\,(1-\chi)\, A_t \left(\frac{K_{t-1}}{L_t}\right)^{\chi}$$

In order to see what effects these have we need an expression for the total profit of production firms per period. The nominal dividend per period is (using the definitions of the real wage, real rental rate for capital and real dividend as defined in the household problem and the actuarial firm problem, respectively)

$$P_t div_t = P_{H,t} Y_t - P_t w_t L_t - P_t r_t^k K_t - \frac{\vartheta}{2}\left(\pi_{H,t}-1\right)^2 P_{H,t}$$

which we can write in real terms

$$div_t = \frac{Y_t}{g(S_t)} - w_t L_t - r_t^k K_t - \frac{\vartheta}{2}\left(\pi_{H,t}-1\right)^2 \frac{1}{g(S_t)}$$

In this case production firms choose prices, capital and labour to maximise profits. That is, the firm sets the price and the quantity adjusts to clear the market.

7.1 Production firms block

Optimal behaviour by production firms is described by the following equations:

Real output

$$Y_t = A_t K_{t-1}^{\chi} L_t^{1-\chi} \tag{7.1}$$

Labour demand

$$MC_t^r\,(1-\chi)\, A_t K_{t-1}^{\chi} L_t^{-\chi} = g(S_t)\, w_t \tag{7.2}$$

Capital demand

$$MC_t^r \chi A_t K_{t-1}^{\chi-1} L_t^{1-\chi} = g(S_t)\, r_t^k \tag{7.3}$$

Definition of real dividend

$$div_t = \frac{Y_t}{g(S_t)} - w_t L_t - r_t^k K_{t-1} - \frac{PCosts_t}{g(S_t)} \tag{7.4}$$

Price adjustment costs

$$PCosts_t = \frac{\vartheta}{2}\left(\pi_{H,t}-1\right)^2 \tag{7.5}$$

Optimal price

$$\varepsilon\vartheta\pi_{H,t}\left(\pi_{H,t}-1\right)=\frac{1}{n}\Upsilon_{D,t}\left(MC_t^r-(1-\varepsilon)\right)+\frac{\varepsilon\vartheta}{1+r_t}\frac{g\left(S_t\right)}{g\left(S_{t+1}\right)}\pi_{H,t+1}\left(\pi_{H,t+1}-1\right) \tag{7.6}$$

Definition of total demand for domestic goods

$$\Upsilon_{D,t}=(1-(1-n)\alpha)\left(\frac{P_t}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t+n\alpha\left(\frac{P_t^*}{P_{H,t}}\right)^{\frac{1}{\eta}}\Upsilon_t^*+\frac{\vartheta}{2}\left(\pi_{H,t}-1\right)^2 \tag{7.7}$$

Domestic demand

$$\Upsilon_t=C_t+I_t+G_t+\Psi\left(\frac{I_t}{K_{t-1}}\right)I_t+\xi\left(\Delta FB_t\right) \tag{7.8}$$

8 Technology and definitions

8.1 Technology and definitions

To complete the model we have various definitions and exogenous processes Technology

$$A_t = A_0 + e_t \tag{8.1}$$

Aggregrate resource constraint

$$Y_t = g(S_t)(\Upsilon_t + NX_t) + PCosts_t \tag{8.2}$$

Net interest definition

$$ni_t = (1 - \tau_{i,t})i_{t-1} \tag{8.3}$$

Household real interest definition

$$1 + r_t^h = \frac{1 + ni_{t+1}}{\pi_{t+1} d} \tag{8.4}$$

Real interest definition

$$1 + r_t = \frac{1 + i_t}{\pi_{t+1}} \tag{8.5}$$

Real foreign interest definition

$$1 + r_t^{fo} = \frac{1 + i_t^{fo}}{\pi_{t+1}} \tag{8.6}$$

9 The complete model

9.1 Household block

Optimal consumption

$$C_t = (1 - s_t) H_t \tag{9.1}$$

Total wealth (note how we have changed the time subscript for financial wealth without consequence)

$$H_t = HW_t + FW_t \tag{9.2}$$

Human wealth

$$HW_t = \begin{array}{l} (1-\tau_{l,t}) w_t L_t + (1-\tau_{i,t}) div_t^{AI} - \frac{1}{1-d} \tau_{ls,t} \\ - \frac{i_{t-1}(1-\tau_{i,t})}{1+i_{t-1}(1-\tau_{i,t})} M_t + \frac{d\pi_{t+1}}{1+i_t(1-\tau_{i,t+1})} HW_{t+1} \end{array} \tag{9.3}$$

Financial wealth

$$FW_t = \frac{1}{\pi_t d} \{[1 + ni_t] N_{t-1} + M_{t-1}\} \tag{9.4}$$

Marginal propensity to consume

$$(1 - s_t)^{-1} = Z_t X_t + \frac{Z_t}{Z_{t+1}} (\beta d)^{\frac{1}{\theta}} \left(1 + r_t^h\right)^{\frac{1}{\theta} - 1} (1 - s_{t+1})^{-1} \tag{9.5}$$

where

$$X_t = \left[\frac{(1 + \tau_{c,t})^{-\varsigma\varphi}}{(w_t (1 - \tau_{l,t}))^{(1-\varphi)}} \left(\frac{1 + ni_{t+1}}{ni_{t+1}} \right)^{\varphi(1-\varsigma)} \right]^{\frac{(1-\theta)}{\theta}} \tag{9.6}$$

and

$$Z_t = \left[\frac{(1 + \tau_{c,t})^{-\varsigma\varphi - \theta/(1-\theta)}}{(w_t (1 - \tau_{l,t}))^{(1-\varphi)}} \left(\frac{1 + ni_{t+1}}{ni_{t+1}} \right)^{\varphi(1-\varsigma)} \right]^{-\frac{(1-\theta)}{\theta}} \tag{9.7}$$

Labour supply

$$L_t = \frac{1}{1-d} - \frac{(1-\varphi)}{\varsigma\varphi} \frac{(1 + \tau_{c,t})}{(1 - \tau_{l,t})} \frac{(1 - s_t)}{w_t} H_t \tag{9.8}$$

Money demand

$$M_t = (1 + \tau_{c,t}) \frac{(1-\varsigma)}{\varsigma} \frac{1 + ni_{t+1}}{ni_{t+1}} (1 - s_t) H_t \tag{9.9}$$

9.2 Actuarial firms block

Demand for production shares

$$1+r_t = \frac{q_{t+1}+div_{t+1}}{q_t} \tag{9.10}$$

Demand for foreign investment firm shares

$$1+r_t = \frac{q\,f_{t+1}+div\,f_{t+1}}{q\,f_t} \tag{9.11}$$

Optimal investment (1)

$$\Lambda_t = 1+\Psi\left(\frac{I_t}{K_{t-1}}-\delta\right)+\frac{I_t}{K_{t-1}}\Psi'\left(\frac{I_t}{K_{t-1}}-\delta\right) \tag{9.12}$$

Optimal investment (2)

$$\Lambda_t = \frac{1}{1+r_t}\left[r^k_{t+1}+\left(\frac{I_{t+1}}{K_t}\right)^2\Psi'\left(\frac{I_{t+1}}{K_t}-\delta\right)+\Lambda_{t+1}(1-\delta)\right] \tag{9.13}$$

Dividend

$$\begin{aligned} div^{AI}_t = \; & N_t-(1+r_{t-1})N_{t-1}-B_t+(1+r_{t-1})B_{t-1}+div_t+div\,f_t \\ & +r^k_t K_{t-1}-\left[1+\psi\left(\tfrac{I_t}{K_{t-1}}-\delta\right)\right]I_t \end{aligned} \tag{9.14}$$

Zero expected profit condition

$$\begin{aligned} 0 = \; & N_{t+1}-(1+r_t)N_t-B_{t+1}+(1+r_t)B_t+div_{t+1}+div\,f_{t+1} \\ & +r^k_{t+1}K_t-\left[1+\psi\left(\tfrac{I_{t+1}}{K_t}-\delta\right)\right]I_{t+1} \end{aligned} \tag{9.15}$$

Definition of investment

$$I_t = K_t-(1-\delta)K_{t-1} \tag{9.16}$$

Adjustment cost function

$$\Psi\left(\frac{I_t}{K_{t-1}}-\delta\right) = cp\times\left(\frac{I_t}{K_{t-1}}-\delta\right)^2 \tag{9.17}$$

9.3 Foreign investment firms block

Optimal foreign bond holding (1)

$$\Lambda f_t = 1+\xi'(\Delta FB_t) \tag{9.18}$$

Optimal foreign bond holding (2)

$$\Lambda f_t = \frac{1}{1+r_t}\left(r^{fo}_t+\Lambda f_{t+1}\right) \tag{9.19}$$

Definition of real dividend

$$div f_t = r^{fo}_{t-1} FB_{t-1} - \Delta FB_t - \xi(\Delta FB_t) \tag{9.20}$$

Definition of foreign bond adjustment

$$\Delta FB_t = FB_t - FB_{t-1} \tag{9.21}$$

Adjustment cost function

$$\xi(\Delta FB_t) = cpf \times (\Delta FB_t)^2 \tag{9.22}$$

Real net exports definition: Capital account

$$NX_t = FB_t - \left(1 + r^{fo}_{t-1}\right) FB_{t-1} \tag{9.23}$$

9.4 Government block

Government spending

$$G_t = G_0 + e^g_t \tag{9.24}$$

Government budget constraint

$$B_t = G_t + (1 + r_{t-1}) B_{t-1} - \tau_{i,t} \frac{i_{t-1}}{\pi_t} N_{t-1} - \tau_{l,t} w_t L_t - \tau_{c,t} C_t - \frac{1}{1-d} \tau_{ls,t} + \tau_{i,t} P_t div^{AI}_t \tag{9.25}$$

Fiscal policy rule

$$\frac{1}{1-d} \tau_{ls,t} = (1 + r_{t-1} + \tau_{sus}) B_{t-1} \tag{9.26}$$

9.5 Aggregators block

Evolution of terms of trade

$$\frac{S_t}{S_{t-1}} = \frac{\pi_{F,t}}{\pi_{H,t}} \tag{9.27}$$

The domestic CPI to PPI ratio

$$g(S_t) = \left[(1 - (1-n)\alpha) + (1-n)\alpha S_t^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{9.28}$$

The foreign CPI to PPI ratio

$$g^*(S_t) = \left[(1 - n\alpha) + n\alpha S_t^{\frac{1-\eta}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{9.29}$$

Real net exports definition: Current account

$$NX_t = \frac{1}{g(S_t)} n\alpha \left(g^*(S_t)S_t\right)^{\frac{1}{\eta}} \Upsilon_t^* - (1-n)\alpha \left(\frac{g(S_t)}{S_t}\right)^{\frac{1-\eta}{\eta}} \Upsilon_t \tag{9.30}$$

Domestic CPI definition

$$\pi_t = \frac{g(S_t)}{g(S_{t-1})} \pi_{H,t} \tag{9.31}$$

9.6 Production firms block

Real output

$$Y_t = A_t K_{t-1}^{\chi} L_t^{1-\chi} \tag{9.32}$$

Labour demand

$$MC_t^r (1-\chi) A_t K_{t-1}^{\chi} L_t^{-\chi} = g(S_t) w_t \tag{9.33}$$

Capital demand

$$MC_t^r \chi A_t K_{t-1}^{\chi-1} L_t^{1-\chi} = g(S_t) r_t^k \tag{9.34}$$

Definition of real dividend

$$div_t = \frac{Y_t}{g(S_t)} - w_t L_t - r_t^k K_{t-1} - \frac{PCosts_t}{g(S_t)} \tag{9.35}$$

Price adjustment costs

$$PCosts_t = \frac{\vartheta}{2} (\pi_{H,t} - 1)^2 \tag{9.36}$$

Optimal price

$$\varepsilon \vartheta \pi_{H,t} (\pi_{H,t} - 1) = \frac{1}{n} \Upsilon_{D,t} (MC_t^r - (1-\varepsilon)) + \frac{\varepsilon \vartheta}{1+r_t} \frac{g(S_t)}{g(S_{t+1})} \pi_{H,t+1} (\pi_{H,t+1} - 1) \tag{9.37}$$

Definition of total demand for domestic goods

$$\Upsilon_{D,t} = (1-(1-n)\alpha) \left(\frac{P_t}{P_{H,t}}\right)^{\frac{1}{\eta}} \Upsilon_t + n\alpha \left(\frac{P_t^*}{P_{H,t}}\right)^{\frac{1}{\eta}} \Upsilon_t^* + \frac{\vartheta}{2} (\pi_{H,t} - 1)^2 \tag{9.38}$$

Domestic demand

$$\Upsilon_t = C_t + I_t + G_t + \Psi \left(\frac{I_t}{K_{t-1}}\right) I_t + \xi (\Delta FB_t) \tag{9.39}$$

9.7 Technology and definitions

Technology

$$A_t = A_0 + e_t \tag{9.40}$$

Aggregrate resource constraint

$$Y_t = g(S_t)(\Upsilon_t + NX_t) + PCosts_t \tag{9.41}$$

Net interest definition

$$ni_t = (1 - \tau_{i,t})i_{t-1} \tag{9.42}$$

Household real interest definition

$$1 + r_t^h = \frac{1 + ni_{t+1}}{\pi_{t+1}d} \tag{9.43}$$

Real interest definition

$$1 + r_t = \frac{1 + i_t}{\pi_{t+1}} \tag{9.44}$$

Real foreign interest definition

$$1 + r_t^{fo} = \frac{1 + i_t^{fo}}{\pi_{t+1}} \tag{9.45}$$

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