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A Financial Market Model for the Netherlands

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Contents

1 Introduction

The Commission Parameters (Langejan et al. (2014)) advises to use the KNW-capital market model to generate a uniform scenario set which enables comparable feasibility tests of pension funds. CPB's task is to estimate the model on Dutch data and to calibrate some parameters to make it consistent with the expectations of the Commission Parameters.

The model describes the stock and bond market (the latter by way of a term structure of interest rates). Both the nominal bonds as well as the development of the stock returns depend on the inflation process which is modelled, too. Net benefits of pensions can be considered as a derivative of bonds and equity, because both the benefits and premiums depend on the investment results. This capital market model is appropriate to evaluate derivative products. The model is developed by Koijen et al. (2010) and estimated by them on US data.

This paper describes estimates based on data relevant for the Netherlands, estimated over a longer period than before (Draper (2012)). It provides a technical documentation of this capital market model and details on the derivations, the estimation and calibration. The estimation results are compared with earlier results.

The Koijen et al. (2010) model is related to Brennan and Xia (2002), Campbell and Viceira (2001) and Sangvinatsos and Wachter (2005). More details of the model can be found in Koijen et al. (2005) and Koijen et al. (2006). A survey of all risks that pension funds are facing, can be found in Broer (2010).

Section 2 presents the model without giving details: its assumptions, parameter restrictions, the link between nominal and inflation linked bonds and the term structure are discussed. Section 3 details on the term structure and on bond funds implementing constant duration. The model is formulated in continuous time, but for simulation and estimation purposes a discretized version is necessary. This discretization and estimation procedure is discussed in section 4. To determine the value of derivative products, as for instance the liabilities of pension funds, risk-neutral simulation is used which is discussed in section 5. The data are presented in section 6. Section 7 gives the estimation results. Section 8 discusses the necessary parameter changes which results in consistency between the model outcomes and the expectations of the Commission Parameters. Section 10 concludes.

2 The Model

2.1 Assumptions

The portfolio consists of a stock index, long-term nominal and real bonds and a nominal money account. The uncertainty and dynamics in the real interest rate and in the instantaneous expected inflation are modelled using two state variables, which are collected in vector X. More precisely, for the instantaneous real interest rate, r, holds

$$r_t = \delta_{0r} + \delta_{1r}' X_t \tag{1}$$

and for the instantaneous expected inflation, π

$$\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t \tag{2}$$

The dynamics in the state variables govern the autocorrelation in the interest rates and inflation. The state variables follow a mean-reverting process around zero[2]

$$dX_t = -KX_t dt + \Sigma_X' dZ_t \tag{3}$$
$$K \text{ is } 2 \times 2 \text{ and } \Sigma_X' = [I_{2\times2} 0_{2\times2}]$$

where Z denotes a four dimensional vector of independent Brownian motions which drive the uncertainty in the financial market. Four sources of uncertainty can be identified: uncertainty about the real interest rate, uncertainty about the instantaneous expected inflation, uncertainty about unexpected inflation and uncertainty about the stock return. Any correlation between the real interest rate and inflation is modelled using δ_{1r}' and $\delta_{1\pi}'$. Expected inflation, π, determines the price index Π:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ_t \quad \sigma_\Pi \in R^4 \text{ and } \Pi_0 = 1 \tag{4}$$

The stock index S develops according to

$$\frac{dS_t}{S_t} = (R_t + \eta_S)dt + \sigma_S' dZ_t \quad \sigma_S \in R^4 \text{ and } S_0 = 1 \tag{5}$$

where R is the nominal instantaneous interest rate, which we determine in the next section and η_S the equity risk premium. The model is completed with the specification of the nominal stochastic discount factor ϕ_t^N

$$\frac{d\phi_t^N}{\phi_t^N} = -R_t dt - \Lambda_t' dZ_t \tag{6}$$

with the time-varying price of risk Λ affine in the state variables X

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t \quad \text{and } \Lambda_t, \Lambda_0 \in R^4 \quad \text{and } \Lambda_1 \; 4\times 2 \tag{7}$$

The stochastic discount factor gives the marginal utility ratio between consumption today and in the future. This marginal utility ratio is for everyone the same in case of complete markets

[2] A thorough book about continous time modelling in Finance is Shreve (2004). Hull (2003) is more convenient to get intuition for the subject.

(everyone can trade in all risks). The stochastic discount factor is thus appropriate to discount all flows. A theoretical justification of this stochastic discount factor can be found in Merton (1992) and Cochrane (2005). The price of risk will depend on the risk aversion of investors. Assume no risk premium for unexpected inflation, i.e. the third row Λ_1 contains zeros only. This restriction is imposed because unexpected inflation risk can't be identified on the basis of data on the nominal side of the economy alone (see Koijen et al. (2010))

$$\Lambda_1 = \begin{bmatrix} \Lambda_{1_{(1,1)}} & \Lambda_{1_{(1,2)}} \\ \Lambda_{1_{(2,1)}} & \Lambda_{1_{(2,2)}} \\ 0 & 0 \\ \Lambda_{1_{(4,1)}} & \Lambda_{1_{(4,2)}} \end{bmatrix} \tag{8}$$

2.2 Some implications

2.2.1 Parameter restrictions

The stochastic discount factor can be used to determine the value of all assets because the described markets are complete. For instance, the fundamental valuation equation (see for instance Cochrane (2005)) of the equity index

$$Ed\phi^N S = 0 \tag{9}$$

implies that the expected value of the discounted stock price does not change over time. This equation implies a restriction. Using the Itô Doeblin theorem gives

$$\begin{aligned} \frac{d\phi^N S}{\phi^N S} &= \frac{d\phi^N}{\phi^N} + \frac{dS}{S} + \frac{d\phi^N}{\phi^N}.\frac{dS}{S} = \\ &= \left(\eta_S - \Lambda_t^{'}\sigma_S^{'}\right)dt - \left(\Lambda_t^{'} - \sigma_S^{'}\right)dZ_t \end{aligned} \tag{10}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. Taking expectations gives the restriction

$$\eta_S = \Lambda_t^{'}\sigma_S \tag{11}$$

which implies $\sigma_S^{'}\Lambda_0 = \eta_S$ and $\sigma_S^{'}\Lambda_1 = 0$. This restriction is imposed on the model.

2.2.2 Nominal and inflation linked bonds

The fundamental pricing equation for a nominal zero coupon bond is

$$Ed\phi^N P^N = 0 \tag{12}$$

i.e. the expected discounted value of the price of a nominal bond does not change over time. The condition implies for inflation linked bonds

$$Ed\phi^N P^R \Pi = 0$$

i.e. the discounted value of the inflation corrected price of real bonds doesn't change over time. Define the real stochastic discount factor as $\phi^R \equiv \phi^N \Pi$. Using the Itô Doeblin theorem we derive for the real stochastic discount factor

$$\begin{aligned}\frac{d\phi^R}{\phi^R} &\equiv \frac{d\phi^N}{\phi^N}+\frac{d\Pi}{\Pi}+\frac{d\phi^N}{\phi^N}\cdot\frac{d\Pi}{\Pi} \\ &= -\left(R_t-\pi_t+\sigma_\Pi'\Lambda_t\right)dt-\left(\Lambda_t'-\sigma_\Pi'\right)dZ_t \\ &= -r_t dt-\left(\Lambda_t'-\sigma_\Pi'\right)dZ_t\end{aligned} \tag{13}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. The nominal rate can thus be written as

$$\begin{aligned}R_t &= r_t+\pi_t-\sigma_\Pi'\Lambda_t \\ &= \left(\delta_{0r}+\delta_{0\pi}-\sigma_\Pi'\Lambda_0\right)+(\delta_{1r}'+\delta_{1\pi}'-\sigma_\Pi'\Lambda_1)X_t \\ &\equiv R_0+R_1'X_t\end{aligned} \tag{14}$$

2.2.3 The nominal and real term structure

Next sections shows that the price of a nominal zero coupon bond which has a single payout at a time $t+\tau$ can be written as

$$P^N(X_t,t,t+\tau)=\exp\left(A^N(\tau)+B^N(\tau)'X_t\right) \tag{15}$$

with A^N and B^N functions of the model parameters. These relations can be used for valuation of the nominal liabilities of pension funds. The next section also shows, that the price of a real zero coupon bond which has a single payout at a time $t+\tau$ can be written as

$$P^R(X_t,t,t+\tau)=\exp\left(A^R(\tau)+B^R(\tau)'X_t\right) \tag{16}$$

with A^R and B^R functions of the model parameters. These relations can be used for valuation of the real liabilities of pension funds.

2.2.4 Valuation of derivatives

The liabilities of pension funds may depend on the capital market returns. The liabilities are then in financial terms derivatives. Two approaches are available to value derivatives in case of complete markets. The first approach constructs a portfolio of stocks and bonds which replicates the returns (pension benefits) of the pension scheme. The determination of this replicating portfolio is very cumbersome and even impossible in practize. The second method uses the stochastic discount factor. This method is very handsome because the stochastic processes of the assets in this continuous time capital market model can be transformed in such a way that the value of all assets can be determined using the nominal instantaneous rate after this transformation. To value derivatives does not need an assessment of the replicating portfolio but can be determined with this risk neutral version of the model. Section 5 details on the risk neutral version of the model.

3 The nominal and real term structure; a digression

3.1 The nominal term structure

A second-order approximation of fundamental pricing equation (12) of a nominal zero coupon bond (thus a single payout at a fixed point in the future) is

$$E\left[d\phi^N.P^N+\phi^N.dP^N+d\phi^N.dP^N\right]=0 \tag{17}$$

Assume bond prices dependent on the state of the economy and a time trend $P^N=P^N(X,t)$. Using the Itô Doeblin theorem we obtain

$$\begin{aligned} dP^N &= P_X^{N\prime}dX+P_t^N dt+\frac{1}{2}dX^{\prime}P_{XX^\prime}^N dX+dX^{\prime}P_{Xt}^N dt+\frac{1}{2}dtP_{tt}^N dt \\ &= P_X^{N\prime}\left(-KX_t dt+\Sigma_X^\prime dZ_t\right)+P_t^N dt+\frac{1}{2}\left(dZ_t\right)\Sigma_X P_{XX^\prime}^N\Sigma_X^\prime dZ_t \end{aligned} \tag{18}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. Substitution of this equation for the price changes and the nominal stochastic discount factor (6) into the fundamental valuation equation (17) brings about

$$0=P_X^{N\prime}\left(-KX_t\right)+P_t^N+\frac{1}{2}tr\left(\Sigma_X P_{XX^\prime}^N\Sigma_X^\prime\right)-P^N R_t-P_X^{N\prime}\Sigma_X^\prime\Lambda_t \tag{19}$$

Note, the trace term (see Cochrane (2005), page 378) appears because only quadratic terms remain due to independence of the error terms. This partial differential equation has a solution of the form

$$P^N(X_t,t,t+\tau)=\exp\left(A^N(\tau)+B^N(\tau)^\prime X_t\right) \tag{20}$$

In case of a single pay-off at time T, duration is defined as $\tau=T-t$. Substitute the derivatives

$$\begin{aligned} &\frac{1}{P^N}P_X^N=B^N \\ &\frac{1}{P^N}P_t^N=-\frac{1}{P^N}P_\tau^N=-\dot{A}^N-\dot{B}^{N\prime}X_t \\ &\frac{1}{P^N}P_{XX^\prime}^N=B^N B^{N\prime} \end{aligned} \tag{21}$$

into the partial differential equation (19)

$$0=B^{N\prime}\left(-KX_t\right)+\left(-\dot{A}^N-\dot{B}^{N\prime}X_t\right)+\frac{1}{2}tr\left(\Sigma_X B^N B^{N\prime}\Sigma_X^\prime\right)-R_0-R_1^\prime X_t-B^{N\prime}\Sigma_X^\prime\left(\Lambda_0+\Lambda_1 X_t\right)$$

(22)

to obtain explicit expressions for A^N and B^N . Note:
$tr\left(\Sigma_X B^N B^{N\prime}\Sigma_X^{\prime}\right) = tr\left(B^{N\prime}\Sigma_X^{\prime}\Sigma_X B^N\right) = B^{N\prime}\Sigma_X^{\prime}\Sigma_X B^N$ because $tr(AB) = tr(BA)$. Both the stochastic term and the non stochastic term have to be zero, leading to

$$\dot{A}^N(\tau) = -R_0 - \left(\Lambda_0^{\prime}\Sigma_X\right) B^N(\tau) + \frac{1}{2} B^{N\prime}(\tau)\Sigma_X^{\prime}\Sigma_X B^N(\tau) \tag{23}$$

$$\dot{B}^N(\tau) = -R_1 - \left(K^{\prime} + \Lambda_1^{\prime}\Sigma_X\right) B^N(\tau) \tag{24}$$

The nominal zero coupon bond with duration $\tau = 0$ and payout 1 has a price $P^N(X_t, t, t) = 1$, which implies $A^N(0) = 0$ and $B^N(0) = 0$. The instantaneous (*i.e.* given the state of the economy) nominal yield of a bond with duration zero (cash) is defined as
$-d\ln P^N(X_t, t, t) = -\left(\dot{A}^N(0) + \dot{B}^N(0)^{\prime} X_t\right) = R_0 + R_1^{\prime} X_t \equiv R_t$. The instantaneous nominal yield of a bond with duration τ is $-d\ln P^N(X_t, t, t+\tau) = -\left(\dot{A}^N(\tau) + \dot{B}^N(\tau)^{\prime} X_t\right)$.. The differential equations can be solved in closed form

$$B^N(\tau) = \left(K^{\prime} + \Lambda_1^{\prime}\Sigma_X\right)^{-1}\left[\exp\left(-\left(K^{\prime} + \Lambda_1^{\prime}\Sigma_X\right)\tau\right) - I_{2\times 2}\right] R_1 \tag{25}$$

$$A^N(\tau) = \int_0^{\tau} \dot{A}^N(s)ds \tag{26}$$

with $I_{2\times 2}$ the two by two identity matrix. These relations will be used for market conform discounting of nominal liabilities of pension funds.

3.2 The real term structure

The fundamental pricing equation of a real zero coupon bond (thus a single payout at a fixed point in the future) is

$$Ed\phi^R P^R = 0 \tag{27}$$

leading to the partial differential equation

$$0 = P_X^{R\prime}(-KX_t) + P_t^R + \frac{1}{2} tr\left(\Sigma_X P_{XX^{\prime}}^R \Sigma_X^{\prime}\right) - P^R r_t - P_X^{R\prime}\Sigma_X^{\prime}\left(\Lambda_t - \sigma_\Pi\right) \tag{28}$$

This partial differential equation has a solution of the form

$$P^R(X_t, t, t+\tau) = \exp\left(A^R(\tau) + B^R(\tau)^{\prime} X_t\right) \tag{29}$$

in case of a single pay-off at time time $t+\tau$. Substitute the derivatives into the fundamental pricing equations leads to

$$0 = B^{R\prime}\left(-KX_t\right) + \left(-\dot{A}^R - \dot{B}^{R\prime}X_t\right) + \frac{1}{2}B^{R\prime}\Sigma_X'\Sigma_X B^R - \left(\delta_{0r} + \delta_{1r}'X_t\right) - B^{R\prime}\Sigma_X'\left(\Lambda_0 - \sigma_\Pi + \Lambda_1 X_t\right) \tag{30}$$

Both the stochastic term and the non stochastic term have to be zero leading to

$$\dot{A}^R = -\delta_{0r} - \left[\left(\Lambda_0' - \sigma_\Pi'\right)\Sigma_X\right]B^R + \frac{1}{2}B^{R\prime}\Sigma_X'\Sigma_X B^R \tag{31}$$

$$\dot{B}^R = -\delta_{1r} - \left(K' + \Lambda_1'\Sigma_X\right)B^R \tag{32}$$

The real zero coupon bond with duration $\tau = 0$ and payout 1 has a price $P^R(X_t,t,t) = 1$, which implies $A^R(0) = 0$ and $B^R(0) = 0$. The instantaneous real yield of cash is $-d\ln P^R(X_t,t,t) = -\left(\dot{A}^R(0) + \dot{B}^R(0)'X_t\right) = \delta_{0r} + \delta_{1r}'X_t \equiv r_t$ and of a bond with duration τ is $-d\ln P^R(X_t,t,t+\tau) = -\left(\dot{A}^R(\tau) + \dot{B}^R(\tau)'X_t\right)$. These relations will be used for market discounting of real liabilities of pension funds.

3.3 Bond funds implementing constant duration

This model will be estimated using yields of bonds with different duration. The introduction of bond funds which implement constant duration is convenient to calculate these yields. This section follows Shi and Werker (2011) and Bajeux-Besnainou et al. (2003). Assume, a bond fund manager rebalances the portfolio permanently to hold the maturity τ constant, i.e. the fund invests only in bonds with maturity τ. The development of the bond index of such a fund can be derived by applying the Itô-Doeblin lemma to[3]

$$P^{F\tau}(X_t) = P^N(X_t,t,\tau) = \exp\left(A^N(\tau) + B^N(\tau)'X_t\right) \tag{33}$$

holding τ constant leads to

$$\begin{aligned} dP^{F\tau} &= P_X^{F\tau\prime}dX + P_t^{F\tau}dt + \frac{1}{2}dX'P_{XX'}^{F\tau}dX + dX'P_{Xt}^{F\tau}dt + \frac{1}{2}dtP_{tt}^{F\tau}dt \\ &= P^{F\tau}B^N(\tau)'dX + \frac{1}{2}P^{F\tau}dX'B^N B^{N\prime}(\tau)dX \end{aligned} \tag{34}$$

After substitution of the state equation (3) and using the Itô Doeblin theorem brings about

$$\frac{dP^{F\tau}}{P^{F\tau}} = \left(-B^N(\tau)'KX_t + \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N\right)dt + B^N(\tau)'\Sigma_X'dZ_t \tag{35}$$

---

[3] Note, the fund's value index can not be determined using the instantaneous return of a bond with constant maturity. The instantaneous return does not take into account changes in the state variables.

This equation together with stochastic discount factor (6) are consistent with the fundamental asset valuation equation if

$$E\left[\frac{dP^{F\tau}}{P^{F\tau}}+\frac{d\phi_t^N}{\phi_t^N}+\frac{dP^{F\tau}}{P^{F\tau}}\frac{d\phi_t^N}{\phi_t^N}\right]=0 \tag{36}$$

which yields the restriction

$$-B^N(\tau)'KX_t+\frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N-R_t-B^N(\tau)'\Sigma_X'\Lambda_t=0 \tag{37}$$

Substitution into equation (35) leads to the funds price dynamics equation

$$\frac{dP^{F\tau}}{P^{F\tau}}=\left(R_t+B^N(\tau)'\Sigma_X'\Lambda_t\right)dt+B^N(\tau)'\Sigma_X'dZ_t \tag{38}$$

This expression has a clear cut interpretation: the dt term is the nominal instantenosu rate plus the risk exposure ($B^N(\tau)'\Sigma_X'$) mutiliplied with the price-of-risk (Λ_t). Note $B^N(0)=0$ leading to $\frac{dP^{F0}}{P^{F0}}=R_t dt$. These relations will be used to construct the portfolio return of pension funds.

4 Estimation procedure

Returns over different time periods are available to estimate the model. This section derives the implications of the model for different time periods, *i.e.* a discrete version of the model is derived.

4.1 Exact discretization

Exact discretization is possible by writing the whole model as a multivariate Ornstein-Uhlenbeck process

$$dY_t=(\Theta_0+\Theta_1 Y_t)dt+\Sigma_Y dZ_t \tag{39}$$

with

$$Y'=\left[\begin{array}{ccccc} X & \ln\Pi & \ln S & \ln P^{F0} & \ln P^{F\tau}\end{array}\right]$$

in which X is the vector with the two state variables, Π the price index, S the stock index, P^{F0} the cash wealth index, $P^{F\tau}$ the bond wealth index with a duration τ, and Z the vector with the four independent Brownian motions extended with two zeros for cash and bond equations. Use Itô Doeblin theorem for log inflation

$$\begin{aligned} d\ln\Pi &= \frac{\partial\ln\Pi}{\partial\Pi}d\Pi+\frac{1}{2}\left(\frac{\partial^2\ln\Pi}{\partial\Pi^2}\right)(d\Pi)^2 \\ &= (\pi_t dt+\sigma_\Pi' dZ_t)-\frac{1}{2}\left[\pi_t dt+\sigma_\Pi' dZ_t\right]^2 \\ &= (\pi_t-\frac{1}{2}\sigma_\Pi'\sigma_\Pi)dt+\sigma_\Pi' dZ_t \end{aligned} \tag{40}$$

and log equity

$$
\begin{aligned}
d\ln S &= \frac{\partial \ln S}{\partial S} dS + \frac{1}{2}\left(\frac{\partial^2 \ln S}{\partial S^2}\right)(dS)^2 \\
&= (R_t + \eta_S)dt + \sigma_S' dZ_t - \frac{1}{2}\left[(r_t + \eta_S)dt + \sigma_S' dZ_t\right]^2 \\
&= (R_0 + R_1' X_t + \eta_S - \frac{1}{2}\sigma_S'\sigma_S)dt + \sigma_S' dZ_t
\end{aligned}
\tag{41}
$$

Log wealth invested in a constant duration fund develops according to

$$
\begin{aligned}
d\ln P^{F\tau} &= \frac{\partial \ln P^{F\tau}}{\partial P^{F\tau}} dP^{F\tau} + \frac{1}{2}\left(\frac{\partial^2 \ln P^{F\tau}}{(\partial P^{F\tau})^2}\right)\left(dP^{F\tau}\right)^2 \\
&= \left(R_t + B^N(\tau)' \Sigma_X' \Lambda_t - \frac{1}{2} B^{N\prime} \Sigma_X' \Sigma_X B^N\right) dt + B^N(\tau)' \Sigma_X' dZ_t
\end{aligned}
$$

This implies for the multivariate Ornstein-Uhlenbeck process

$$
\begin{aligned}
d\begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} &= \left(\begin{bmatrix} 0 \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ R_0 + \eta_S - \frac{1}{2}\sigma_S'\sigma_S \\ R_0 \\ R_0 + B^N(\tau)'\Sigma_X'\Lambda_0 - \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N \end{bmatrix} + \begin{bmatrix} -K & 0 \\ \delta_{1\pi}' & 0 \\ R_1' & 0 \\ R_1' & 0 \\ R_1' + B^N(\tau)'\Sigma_X'\Lambda_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix}\right) dt \\
&\quad + \begin{bmatrix} \Sigma_X' \\ \sigma_\Pi' \\ \sigma_S' \\ 0 \\ B^N(\tau)'\Sigma_X' \end{bmatrix} dZ_t
\end{aligned}
\tag{42}
$$

After using the eigenvalue decomposition

$$
\Theta_1 = UDU^{-1} \tag{43}
$$

the exact discretization reads as

$$
Y_{t+h} = \mu^{(h)} + \Gamma^{(h)} Y_t + \varepsilon_{t+h} \quad \text{and} \quad \varepsilon_{t+h} \sim N(0, \Sigma^{(h)}) \tag{44}
$$

in which:

(ι.) $\Gamma^{(h)}$ is defined as

$$
\Gamma^{(h)} = \exp(\Theta_1 h) = U \exp(Dh) U^{-1} \tag{45}
$$

The matrix exponential is defined as

$$
\exp(A) = I + \sum_{r=1}^{\infty} \frac{1}{r!} A^r \tag{46}
$$

($\iota\iota$.) $\mu^{(h)}$ is defined as

$$\mu^{(h)} = UFU^{-1}\Theta_0 \tag{47}$$

where F a diagonal matrix with elements

$$F_{ii} = h\alpha\left(D_{ii}h\right) \tag{48}$$

with

$$\alpha(x) = \frac{\exp(x)-1}{x} \quad \text{and} \quad \alpha(0) = 1 \tag{49}$$

($\iota\iota\iota$.) $\Sigma^{(h)}$ is defined as

$$\Sigma^{(h)} = UVU' \tag{50}$$

with

$$V_{ij} = \left[U^{-1}\Sigma_Y\Sigma_Y'(U^{-1})'\right]_{ij} h\alpha\left(\left[D_{ii}+D_{jj}\right]h\right) \tag{51}$$

These relations are taken from Koijen et al. (2005) and Bergstrom (1984).

4.2 Restrictions on expected values and volatilities

This section gives expressions for the long- and short-term expected values and volatilities which can be used to impose restrictions. The expected long-run inflation, equity- and bond-returns are the expected values after convergence of the state to zero are

$$\begin{bmatrix} E\frac{d\Pi}{\Pi} \\ E\frac{dS}{S} \\ E\frac{dP^{F0}}{P^{F0}} \\ E\frac{dP^{F\tau}}{P^{F\tau}} \end{bmatrix} = \begin{bmatrix} \delta_{0\pi} \\ R_0+\eta_S \\ R_0 \\ R_0+B^N(\tau)'\Sigma_X'\Lambda_0 \end{bmatrix}$$

The long-run volatilities are defined in the same way

$$\begin{bmatrix} E\left(\frac{d\Pi}{\Pi}-E\frac{d\Pi}{\Pi}\right)^2 \\ E\left(\frac{dS}{S}-E\frac{dS}{S}\right)^2 \\ E\left(\frac{dP^{F0}}{P^{F0}}-E\frac{dP^{F0}}{P^{F0}}\right)^2 \\ E\left(\frac{dP^{F\tau}}{P^{F\tau}}-E\frac{dP^{F\tau}}{P^{F\tau}}\right)^2 \end{bmatrix} = \begin{bmatrix} \sigma_\Pi'\sigma_\Pi \\ \sigma_S'\sigma_S \\ 0 \\ B^{N\prime}\Sigma_X'\Sigma_X B^N \end{bmatrix}$$

The short-term may deviate from these long term values through the dynamics in the state variables. Define $\widetilde{Y}' = \left[X, \Delta\ln\Pi, \Delta\ln S, \Delta\ln P^{F0}, \Delta\ln P^{F\tau}\right]$ and write equation (44) for $h=1$ as

$$\widetilde{Y}_{t+1} = \mu + \Gamma\widetilde{Y}_t + \varepsilon_{t+1} \quad \text{and} \quad \varepsilon_t \sim N(0,\Sigma) \tag{52}$$

Note the columns of Γ linked to inflation and the equity and bond returns are zero. The eigenvalues of Γ lie inside the unit circle, which implies for the long run expected value of $\widetilde{Y}$

$$E\left(\widetilde{Y}\right) = [I - \Gamma]^{-1}\mu \tag{53}$$

and the long-run variance covariance matrix

$$V(\widetilde{Y}) = (I - \Gamma)^{-1}\Sigma\left(I - \Gamma'\right)^{-1} \tag{54}$$

4.3 Likelihood

Assume, two yields are observed without measurement error. For those yields hold

$$y_t^\tau = \left(-A(\tau) - B(\tau)' X_t\right)/\tau \tag{55}$$

These observations can be used to determine the state vector X, given a set parameters which determine A and B. The other four yields are observed with a measurement error by assumption.

$$y_t^\tau = \left(-A(\tau) - B(\tau)' X_t\right)/\tau + \upsilon_t^\tau \quad \text{and} \quad \upsilon_t' \sim N(0, \Sigma^\tau) \tag{56}$$

with $\upsilon_t' = \left[\upsilon_t^{\tau 1}, \upsilon_t^{\tau 2}, \upsilon_t^{\tau 3}, \upsilon_t^{\tau 4}\right]$ Assume no correlation between the measurement errors. This system of measurement equations is extended with the equations from (52) for inflation and equity. The relevant part of the error term extended with zero's is $\widetilde{\varepsilon}$. The quasi log likelihood

$$\ln L = -0.5\left(T\ln|\Sigma^\tau| - \sum_{t=1}^{T}\upsilon_t\left(\Sigma^\tau\right)^{-1}\upsilon_t'\right) - 0.5\left(T\ln|\Sigma| - \sum_{t=1}^{T}\widetilde{\varepsilon}_t\left(\Sigma\right)^{-1}\widetilde{\varepsilon}_t'\right) - 0.5T\ln|B| \tag{57}$$

with $B' = [B(\tau 5), B(\tau 6)]$ is maximized with respect to the parameters using the method of simulated annealing of Goffe et al. (1994) to find the global optimum. Parameter restrictions can be imposed eventually. Duffee (2002) details on the construction of this quasi log likelihood.

5 Risk neutral simulation

Equation set (42) together with stochastic discount factor (6) are consistent with the fundamental asset valuation equations, *i.e.* the expected discounted value of the price of an asset does not change over time. However, to value derivative assets exact knowledge of the risk exposure, *i.e.* the replicating portfolio, will be necessary. The determination of this replicating portfolio is very cumbersome and often even impossible. However, the system of equations[4]

$$d\begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} = \left(\begin{bmatrix} -\Sigma_X'\Lambda_0 \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ R_0 - \frac{1}{2}\sigma_S'\sigma_S \\ R_0 \\ R_0 - \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N \end{bmatrix} + \begin{bmatrix} -\left(K+\Sigma_X'\Lambda_1\right) & 0 \\ \delta_{1\pi}' & 0 \\ R_1' & 0 \\ R_1' & 0 \\ R_1' & 0 \end{bmatrix} \begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} \right) dt$$

$$+ \begin{bmatrix} \Sigma_X' \\ \sigma_\Pi' \\ \sigma_S' \\ 0 \\ B^N(\tau)'\Sigma_X' \end{bmatrix} d\widetilde{Z}_t \tag{58}$$

together with the stochastic discount factor[5]

$$\frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N} = -R_t dt \tag{59}$$

is consistent with the fundamental asset valuation equation, too. This makes the valuation of derivative products as net pension benefits easy because the portfolio composition of the pension fund becomes irrelevant for the determination of the discount factor in this risk-neutral setting. Indeed the discount factor is for all assets equal to R_t in the risk neutral setting. In summary: stochastic process (58) and discount factor (59) lead to the same expected value of the basic assets as (42) together with stochastic discount factor (6) starting from the values of the basic assets in a base year. But, stochastic process (58) and discount factor (59) are most convenient for the valuation of derivatives. Next subsections detail on the different equations of system (58).

5.1 Risk-neutral simulation and the term structure

The state equations (3) together with the stochastic discount factor (6) and fundamental pricing equation (12) bring about partial differential equation (19). Some reordering results in

$$0 = P_X^{N\prime}\left[\left(-\Sigma_X'\Lambda_0\right) - \left(K+\Sigma_X'\Lambda_1\right)X_t\right] + P_t^N + \frac{1}{2}tr\left(\Sigma_X P_{XX'}^N \Sigma_X'\right) - P^N R_t \tag{60}$$

---

[4] The risk premium for equity, η_S, and bonds, $B^N(\tau)'\Sigma_X'\Lambda_t$, are not included in respectively the equity and bond fund equation of system (58), contrary to system (42). Moreover, the constant and slope coefficients of the state equations are different in both systems. The distribution of $\widetilde{Z}$ is again a Brownian motion, but it is another stochast than Z in system (42) in the sense that $\widetilde{Z}$ has another value than Z for every state of the world contrary to the values of X, Π, ... which are the same in (42) and (58).

[5] Note the stochastic discount factor (59) in the risk neutral setting does not include the error terms as in equation (6).

The stochastic process for the state variables (the first row of equation 58)

$$dX_t = \left[\left(-\Sigma_X'\Lambda_0\right) - \left(K + \Sigma_X'\Lambda_1\right)X_t\right]dt + \Sigma_X' dZ_t \tag{61}$$

together with the discount factor (59) and the fundamental valuation equation $Ed\widetilde{\phi}^N P^N = 0$ lead to partial differential equation (60), too, as can easily be proved. The term structure coefficients will be equal to (25) and (26) in this risk neutral setting by deduction. Equation (61) is part of system (58).

5.2 Risk neutral simulation equity

Equity equation (5) together with the stochastic discount factor (6) satisfies the fundamental asset equation (10) because restriction (11) is imposed. Dynamic equity equation

$$\frac{dS_t}{S_t} = R_t dt + \sigma_S' dZ_t \tag{62}$$

and stochastic discount (59) are consistent with the fundamental valuation equation $Ed\widetilde{\phi}^N S = 0$. The log-linear version of equation (62) is part of system (58).

5.3 Risk neutral simulation of bond funds implementing constant duration

Applying the Itô-Doeblin lemma to

$$P^{F\tau}(X_t, t, \tau) = \exp\left(A^N(\tau) + B^N(\tau)' X_t\right) \tag{63}$$

holding τ constant leads to

$$\begin{aligned} dP^{F\tau} &= P_X^{F\tau\prime} dX + P_t^{F\tau} dt + \frac{1}{2} dX' P_{XX'}^{F\tau} dX + dX' P_{Xt}^{F\tau} dt + \frac{1}{2} dt P_{tt}^{F\tau} dt \\ &= P^{F\tau} B^N(\tau)' \left(\left[\left(-\Sigma_X'\Lambda_0\right) - \left(K + \Sigma_X'\Lambda_1\right)X_t\right]dt + \Sigma_X' dZ_t\right) + \frac{1}{2} P^{F\tau} tr\left(\Sigma_X B^N B^{N\prime}(\tau)\Sigma_X'\right) dt \end{aligned} \tag{64}$$

using equation (61). After reordering follows

$$\frac{dP^{F\tau}}{P^{F\tau}} = \left(B^N(\tau)'\left(-\Sigma_X'\Lambda_0\right) - B^N(\tau)'\left(K + \Sigma_X'\Lambda_1\right)X_t + \frac{1}{2} B^{N\prime}\Sigma_X'\Sigma_X B^N\right)dt + B^N(\tau)'\Sigma_X' dZ_t \tag{65}$$

This equation together with the stochastic discount factor (59) are consistent with the fundamental asset valuation equation $Ed\widetilde{\phi}^N P^{F\tau} = 0$ if

$$E\left[\frac{dP^{F\tau}}{P^{F\tau}} + \frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N} + \frac{dP^{F\tau}}{P^{F\tau}}\frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N}\right] = 0 \tag{66}$$

or

$$-B^N(\tau)' KX_t + \frac{1}{2} B^{N\prime}\Sigma_X' \Sigma_X B^N - R_t - B^N(\tau)' \Sigma_X' \Lambda_t = 0 \tag{67}$$

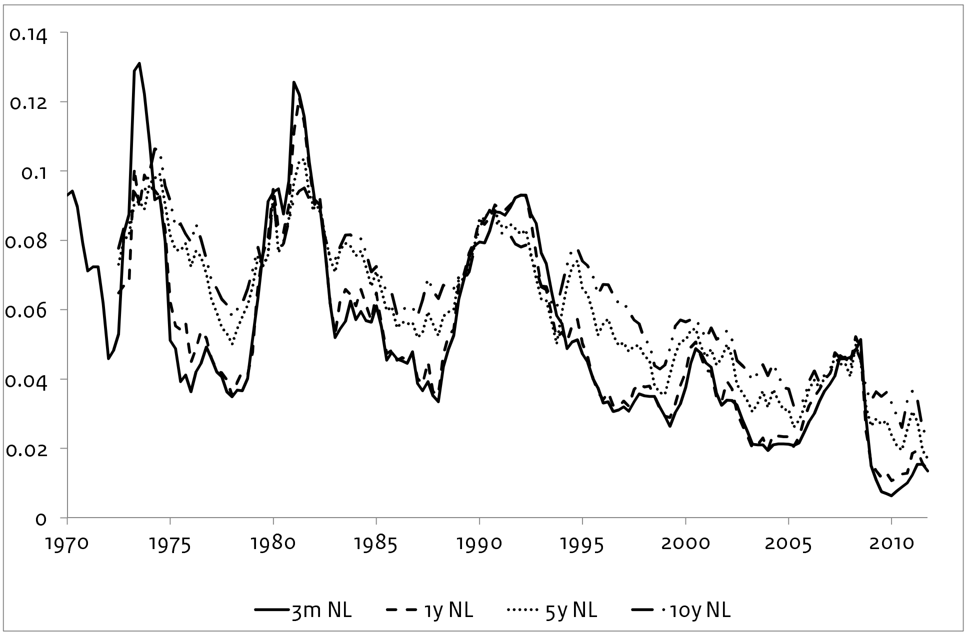
This leads to the conclusion that the funds dynamic equation

$$\frac{dP^{F\tau}}{P^{F\tau}} = R_t dt + B^N(\tau)' \Sigma_X' dZ_t \tag{68}$$

together with the stochastic discount factor (59) lead to the same valuation of assets. The log-linear version of equation (68) is part of system (58).

6 Data

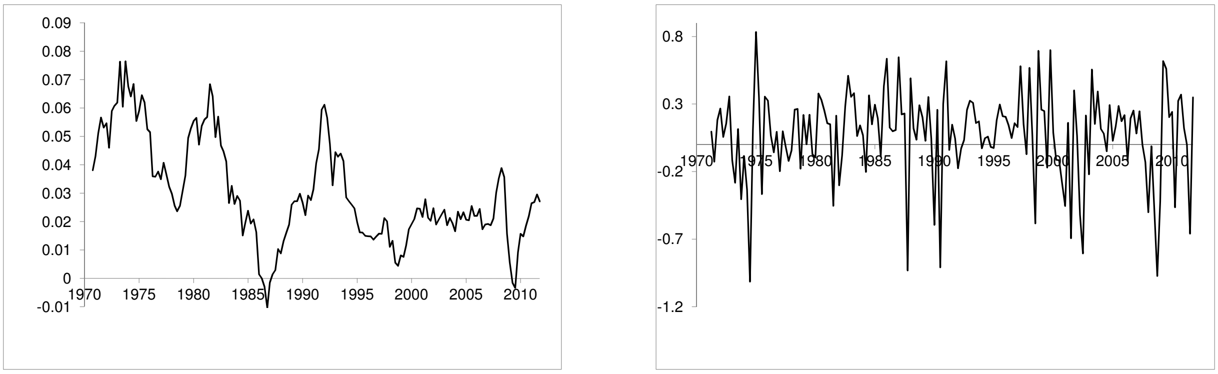
Figure 1 Nominal yields NL duration 3 month, 1, 5 and 10 years



The data for the Netherlands are taken from Goorbergh et al. (2011). All returns are geometric means.

- Inflation: From 1999 on, the Harmonized Index of Consumer Prices for the euro area from the European Central Bank data website (http://sdw.ecb.europa.eu) is used. Before then, German (Western German until 1990) consumer price index figures published by the International Financial Statistics of the International Monetary Fund are included.

Figure 2 Inflation (left) and equity return (right)



- Yields: Six yields are used in estimation: three-month, one-year, two-year, three-year, five-year, and ten-year maturities, respectively. Three-month money market rates are taken from the Bundesbank (www.bundesbank.de). For the period 1973:I to 1990:II, end-of-quarter money market rates reported by Frankfurt banks are taken, whereas thereafter three-month Frankfurt Interbank Offered Rates are included. Long nominal yields: From 1987:IV on, zero-coupon rates are constructed from swap rates published by De Nederlandsche Bank (www.dnb.nl). For the period 1973:I to 1987:III, zerocoupon yields with maturities of one to 15 years (from the Bundesbank website) based on government bonds were used as well (15-year rates start in June 1986). No adjustments were made to correct for possible differences in the credit risk of swaps, on the one hand, and German bonds, on the other. The biggest difference in yield between the two term structures (for the two-year yield) in 1987:IV was only 12 basis points.
- Stock returns: MSCI index from Fact Set. Returns are in euros (Deutschmark before 1999) and hedged for US dollar exposure.

Comments:

- Decreasing bond rates over time after 1980
- In the period 1973-1974, 1980-1981 and 1990-1991 inverse term structure in the Netherlands;
- Inflation for the Netherlands year to year figures.

7 Estimation results

The estimation results are presented in Table 1. The four left columns give the results for a longer estimation period than the two most right columns which are presented earlier in Draper

(2012). Note the sign switch of some significant parameters ($\delta_{1\pi(1)}$, $R_{1(1)}$ and $\Lambda_{0(1)}$). This points to indeterminacy of the sign. Indeed, defining a new state variable with the opposite sign leads to the same maximum likelihood with a different sign for the coefficients. This makes the interpretation of the two estimation results more clear cut. Both specifications are observationally equivalent[6].

The unconditional expected inflation is 1.8% in the Netherlands. Moreover, the persistence is large in the Netherlands and seems to increase. The first-order autocorrelation on an annual frequency equals 0.88 and 0.91 for the real rate and expected inflation, respectively (0.82 and 0.89 with the shorter period). The equity risk premium (η_S) is 4.5% for the Netherlands and seems to have increased. Table 2 reports the risk premium on nominal bonds along with their volatilities. The risk premium for bonds increased a little bit just like their volatility.

[6] The estimates are presented both, because the scenarios are constructed with the first set, while the second set can be compared with the previous estimation results.

Table 1 Estimation results for the Netherlands

Parameter	1972.4-2013.4 Estimate (1)	(SD)	Estimate (2)	(SD)	1972.4-2011.3 Estimate	(SD)
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$						
$\delta_{0\pi}$	1.81%	(2.79%)	1.87%	(2.23%)	2.24%	(1.45%)
$\delta_{1\pi(1)}$	−0.63%	(0.10%)	0.48%	(0.18%)	0.49%	(0.27%)
$\delta_{1\pi(2)}$	0.14%	(0.24%)	0.43%	(0.15%)	0.49%	(0.24%)
Nominal interest rate $R_0 + R_1' X_t$						
R_0	2.40%	(6.06%)	2.53%	(4.86%)	3.70%	(2.77%)
$R_{1(1)}$	−1.48%	(0.22%)	1.29%	(0.32%)	1.40%	(0.43%)
$R_{1(2)}$	0.53%	(0.56%)	0.91%	(0.41%)	0.82%	(0.68%)
Process real interest rate and expected inflation $dX_t = -KX_t dt + \Sigma_X' dZ_t$						
κ_{11}	0.08	(0.11)	0.35	(0.19)	0.32	(0.23)
κ_{22}	0.35	(0.18)	0.08	(0.10)	0.13	(0.13)
κ_{21}	−0.19	(0.08)	−0.20	(0.17)	−0.23	(0.16)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ_t$						
$\sigma_{\Pi(1)}$	0.02%	(0.07%)	−0.02%	(0.07%)	−0.01%	(0.07%)
$\sigma_{\Pi(2)}$	−0.01%	(0.06%)	−0.02%	(0.06%)	−0.01%	(0.06%)
$\sigma_{\Pi(3)}$	0.61%	(0.04%)	0.61%	(0.04%)	0.60%	(0.04%)
Stock return process $\frac{dS_t}{S_t} = (r_t + \eta_S)dt + \sigma_S' dZ_t$						
η_S	4.52%	(3.73%)	4.54%	(3.73%)	3.52%	(3.88%)
$\sigma_{S(1)}$	−0.53%	(1.44%)	−0.32%	(1.59%)	−0.16%	(1.71%)
$\sigma_{S(2)}$	−0.76%	(1.54%)	0.88%	(1.52%)	1.01%	(1.61%)
$\sigma_{S(3)}$	−2.11%	(1.51%)	−2.09%	(1.52%)	−2.65%	(1.56%)
$\sigma_{S(4)}$	16.59%	(0.96%)	16.60%	(0.95%)	16.71%	(0.98%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$						
$\Lambda_{0(1)}$	0.403	(0.333)	−0.200	(0.313)	−0.271	(0.266)
$\Lambda_{0(2)}$	0.039	(0.270)	−0.347	(0.266)	−0.279	0.238)
$\Lambda_{1(1,1)}$	0.149	(0.156)	0.135	(0.218)	0.167	(0.252)
$\Lambda_{1(1,2)}$	−0.381	(0.039)	−0.080	(0.150)	−0.114	(0.239)
$\Lambda_{1(2,1)}$	0.089	(0.075)	0.401	(0.183)	0.395	(0.246)
$\Lambda_{1(2,2)}$	−0.083	(0.129)	−0.068	(0.121)	−0.126	(0.140)

Table 2 Analytically determined long-run risk premia and volatilities

	1972.4-2013.4		1972.4-2011.3	
Maturities	risk premium	volatility	risk premium	volatility
One-year	0.52%	1.33%	0.53%	1.37%
Five-year	1.94%	4.99%	1.80%	5.1%
Ten-year	3.11%	9.10%	2.71%	9.36%

8 Calibration

The Commission Parameters has formulated expectations for some financial variables which are summarized in Table (3). Returns are defined using geometric discounting

Table 3 Adjustments Commission Parameters

Expected equity return	7.0%
Expected price inflation	2.0%
Standard deviation equity return	20.0%
Ultimate forward rate	3.9%

These expectations are generated by the model after calibration of some parameters. These parameter changes are summarized in Table 4. .The parameter that governs the long term

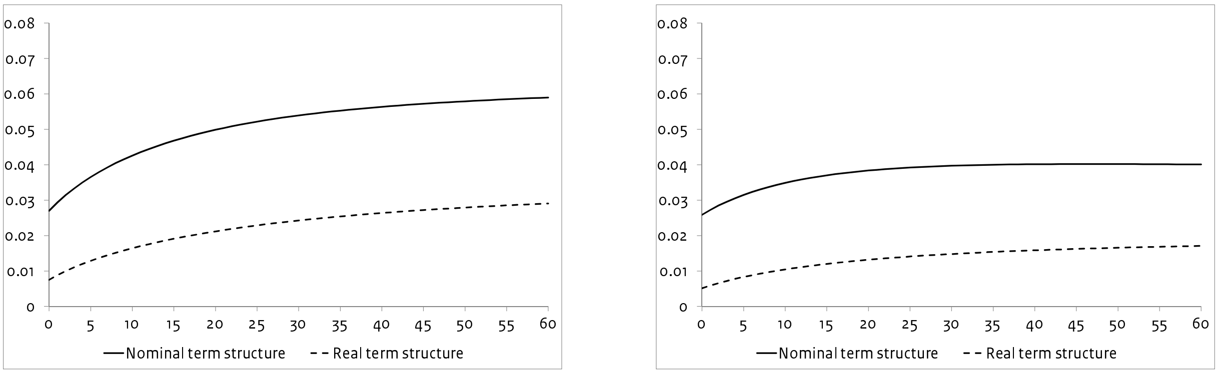
Table 4 Expectations Commission Parameters

Parameter	Estimated	Calibrated
$\delta_{0\pi}$	1.81%	1.98%
η_S	4.52%	6.57%
$\sigma_{S(4)}$	16.59%	17.69%
$\Lambda_{0(1)}$	0.403	0.242

inflation ($\delta_{0\pi}$) is increased a little bit to get an inflation expectation of 2%. One of the parameters that govern the risk premium on bonds ($\Lambda_{0(1)}$) is reduced to obtain a long-run bond return equal to the ultimate forward rate of 3.9%. Lastly the equity risk premium (η_S) and one of the parameters that govern the volatility of equity ($\sigma_{S(4)}$), is increased.

Figure 3 presents the term structure in case the economy is in equilibrium (state variables are then zero). The left panel gives the structure based on the estimation result while the right panel is based on the calibration.

Figure 3 Term structure estimated (left) and calibrated (right)



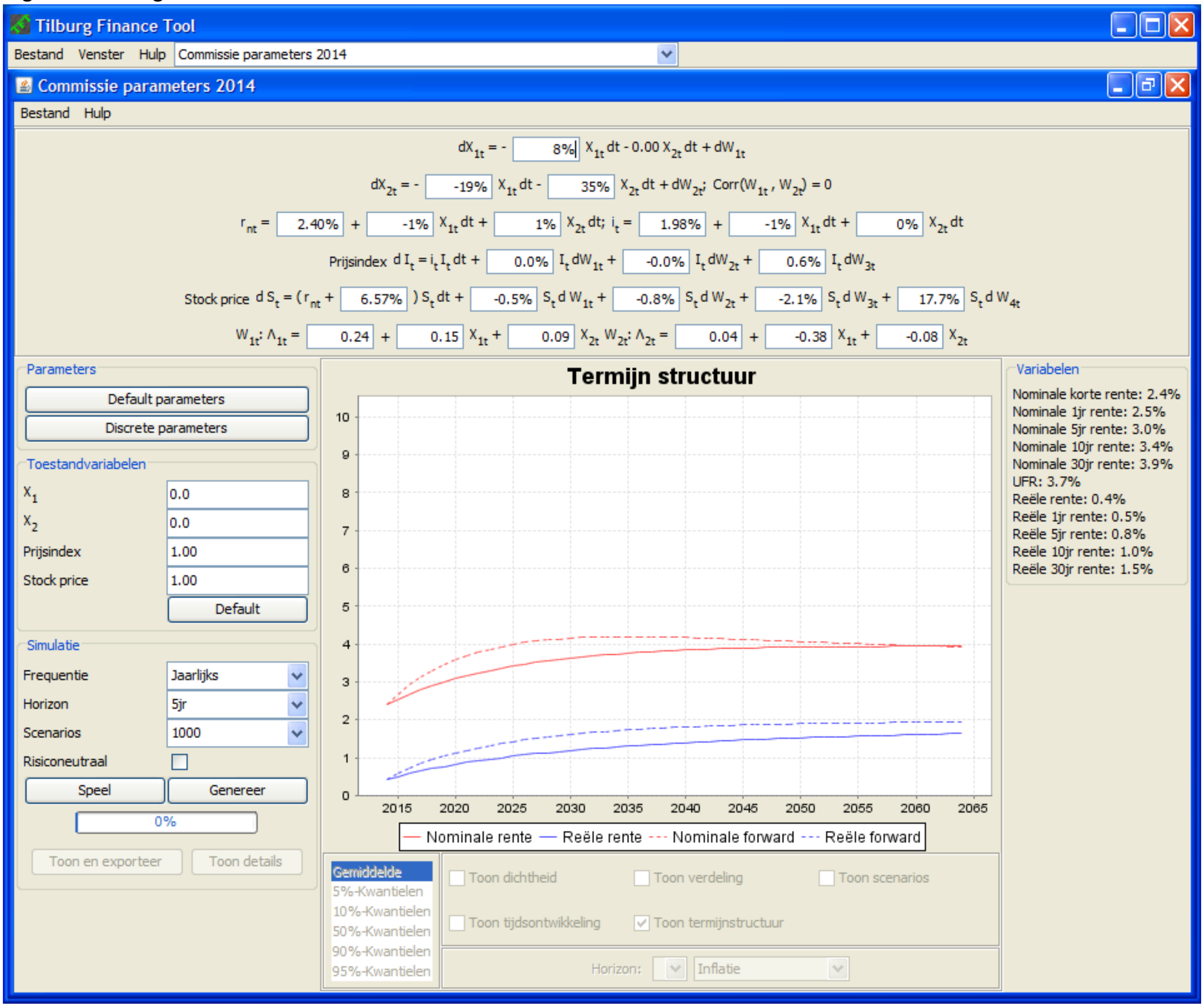
9 Tilburg Finance Tool

Tilburg Finance Tool is an open source analyzer of financial markets and pension funds. The recently released version 1.5 in particular allows simulation of scenarios using the model and parameters prescribed by the Dutch Committee Parameters 2014 (see Figure 4). TFT can be used to gain insight in the (un)certainty in future pensions as well as calculate Holistic Balance Sheets and generational transfers. TFT can be downloaded here.

10 Conclusion

This paper estimates the capital market model of Koijen et al. (2010) using Dutch data. The model does not generate right away the expectations for the coming years of the Commission Parameters. Indeed, the model assumes constant parameters without structural breaks. Some parameters are calibrated after estimation to make the model consistent with the expectations of the Commission Parameters.

Figure 4 Settings TFT consistent with Dutch Committee Parameters 2014



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