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A Financial Market Model for the US and the Netherlands

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Nick Draper[1]

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Contents

1 Introduction

This paper describes a capital market model for the Netherlands. This capital market model will be used, amongst other models, to evaluate the pension agreement between the government, employers and employees (Kamerstukken II 2010/11, 30413, nr. 157) in the Netherlands. Pension funds may invest in a stock market index and in bond funds which rebalance permanently their portfolio to fix the maturity. So we need a description of the stock market and the bond market inclusive the term structure. Both the nominal bonds as well as the development of the stock returns depend on the inflation process which needs to be modelled, too. Net benefits of pensions can be considered as a derivative of bonds and equity, because both the benefits and premiums depend on the investment results. So we need an instrument to evaluate derivative products. The Koijen et al. (2010) model meets all these requirements and will be used after reestimation using Dutch data.

This paper provides a technical documentation of this capital market model and details on the derivations and the estimation. The estimation results are compared to estimates for the US. The results for the Netherlands deviate in several respects from those for the US. In particular, the estimates are less significant. A possible explanation may be that (not included) exchange rate fluctuations are more relevant for Europe than for the US. Due to the small open economy character and the data construction this model can be considered as a model for the north European capital market.

This estimated capital market model will be used to construct scenario sets. These scenarios will be used in an asset liability model (ALM) to evaluate the pension agreement.

The Koijen et al. (2010) model is related to Brennan and Xia (2002), Campbell and Viceira (2001) and Sangvinatsos and Wachter (2005). More details of the model can be found in Koijen et al. (2005) and Koijen et al. (2006). A survey of all risks that pension funds are facing, can be found in Broer (2010).

Section 2 presents the model assumptions. The link between nominal and inflation linked bonds is discussed in section 3, just as the term structure. Bond funds implementing constant duration are the subject of section 4. The model is formulated in continuous time, but for simulation and estimation purposes a discretized version is necessary. This discretization is discussed in section 5. To determine the value of derivative products, as for instance pension rights, risk-neutral simulation is used which is discussed in section 6. The data used to estimate the model are discussed in section 7. Section 8 presents the estimation procedure and section 9 the estimation results. The model will be used to generate scenarios. Section 10 presents some possible scenarios to get some feeling for the results. Section 11 concludes.

2 Model assumptions

The portfolio consists of a stock index, long-term nominal and real bonds and a nominal money account. The uncertainty and dynamics in the real interest rate and in the instantaneous expected inflation are modelled using two state variables, which are collected in vector X. More precisely, for the instantaneous real interest rate, r, holds

$$r_t = \delta_{0r} + \delta'_{1r} X_t \tag{1}$$

and for the instantaneous expected inflation, π

$$\pi_t = \delta_{0\pi} + \delta'_{1\pi} X_t \tag{2}$$

The dynamics in the state variables govern the autocorrelation in the interest rates and inflation. The state variables follow a mean-reverting process around zero[2]

$$dX_t = \mu dt - KX_t dt + \Sigma_X^{'} dZ_t \tag{3}$$
$$K \text{ is } 2 \times 2 \text{ and } \Sigma_X^{'} = [I_{2\times2} 0_{2\times2}]$$

where Z denotes a four dimensional vector of independent Brownian motions which drive the uncertainty in the financial market. Four sources of uncertainty can be identified: uncertainty about the real interest rate, uncertainty about the instantaneous expected inflation, uncertainty about unexpected inflation and uncertainty about the stock return. Any correlation between the real interest rate and inflation is modelled using δ'_{1r} and $\delta'_{1\pi}$. Expected inflation, π, determines the price index Π:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi^{'} dZ_t \quad \sigma_\Pi \in R^4 \text{ and } \Pi_0 = 1 \tag{4}$$

The stock index S develops according to

$$\frac{dS_t}{S_t} = (R_t + \eta_S) dt + \sigma'_S dZ_t \quad \sigma_S \in R^4 \text{ and } S_0 = 1 \tag{5}$$

where R is the nominal instantaneous interest rate, which we determine in the next section and η_S the equity risk premium. The model is completed with the specification of the nominal stochastic discount factor ϕ_t^N

$$\frac{d\phi_t^N}{\phi_t^N} = -R_t dt - \Lambda_t^{'} dZ_t \tag{6}$$

[2] A thorough book about continous time modelling in Finance is Shreve (2004). Hull (2003) is more convenient to get intuition for the subject.

with the time-varying price of risk Λ affine in the state variables X

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t \quad \text{and } \Lambda_t, \Lambda_0 \in R^4 \quad \text{and } \Lambda_1 \; 4 \times 2 \tag{7}$$

To get some intuition, interpret the stochastic discount factor like a marginal utility change. For instance, positive return shocks lead to a decline in marginal utility. A theoretical justification of this stochastic discount factor can be found in Merton (1992) and Cochrane (2005). The price of risk will depend on the risk aversion of investors. Assume no risk premium for unexpected inflation, i.e. the third row Λ_1 contains zeros only. This restriction is imposed because unexpected inflation risk can't be identified on the basis of data on the nominal side of the economy alone (see Koijen et al. (2010))

$$\Lambda_1 = \begin{bmatrix} \Lambda_{1_{(1,1)}} & \Lambda_{1_{(1,2)}} \\ \Lambda_{1_{(2,1)}} & \Lambda_{1_{(2,2)}} \\ 0 & 0 \\ \Lambda_{1_{(4,1)}} & \Lambda_{1_{(4,2)}} \end{bmatrix} \tag{8}$$

The stochastic discount factor can be used to determine the value of all assets in a complete market. For instance, the fundamental valuation equation (see for instance Cochrane (2005)) of the equity index

$$E d\phi^N S = 0 \tag{9}$$

implies that the expected value of the discounted stock price does not change over time. This equation implies a restriction. Using the Itô Doeblin theorem gives

$$\begin{aligned} \frac{d\phi^N S}{\phi^N S} &= \frac{d\phi^N}{\phi^N} + \frac{dS}{S} + \frac{d\phi^N}{\phi^N} \cdot \frac{dS}{S} = \\ &= \left(\eta_S - \Lambda_t' \sigma_S'\right) dt - \left(\Lambda_t' - \sigma_S'\right) dZ_t \end{aligned} \tag{10}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. Taking expectations gives the restriction

$$\eta_S = \Lambda_t' \sigma_S \tag{11}$$

which implies $\sigma_S' \Lambda_0 = \eta_S$ and $\sigma_S' \Lambda_1 = 0$. This restriction is imposed on the model.

3 Nominal and inflation linked bonds

The fundamental pricing equation for a nominal zero coupon bond is

$$E d\phi^N P^N = 0 \tag{12}$$

i.e. the expected discounted value of the price of a nominal bond does not change over time. The no arbitrage condition implies for inflation linked bonds

$$Ed\phi^N P^R \Pi = 0$$

i.e. the discounted value of the inflation corrected price of real bonds doesn't change over time. Define the real stochastic discount factor as $\phi^R \equiv \phi^N \Pi$. Using the Itô Doeblin theorem we derive for the real stochastic discount factor

$$\begin{aligned} \frac{d\phi^R}{\phi^R} &\equiv \frac{d\phi^N}{\phi^N} + \frac{d\Pi}{\Pi} + \frac{d\phi^N}{\phi^N} \cdot \frac{d\Pi}{\Pi} \\ &= -\left(R_t - \pi_t + \sigma_\Pi' \Lambda_t\right) dt - \left(\Lambda_t' - \sigma_\Pi'\right) dZ_t \\ &= -r_t dt - \left(\Lambda_t' - \sigma_\Pi'\right) dZ_t \end{aligned} \tag{13}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. The nominal rate can thus be written as

$$\begin{aligned} R_t &= r_t + \pi_t - \sigma_\Pi' \Lambda_t \\ &= \left(\delta_{0r} + \delta_{0\pi} - \sigma_\Pi' \Lambda_0\right) + (\delta_{1r}' + \delta_{1\pi}' - \sigma_\Pi' \Lambda_1) X_t \\ &\equiv R_0 + R_1' X_t \end{aligned} \tag{14}$$

3.1 The nominal term structure

A second-order approximation of fundamental pricing equation (12) of a nominal zero coupon bond (thus a single payout at a fixed point in the future) is

$$E\left[d\phi^N . P^N + \phi^N . dP^N + d\phi^N . dP^N\right] = 0 \tag{15}$$

Assume bond prices dependent on the state of the economy and a time trend $P^N = P^N(X,t)$. Using the Itô Doeblin theorem we obtain

$$\begin{aligned} dP^N &= P_X^{N\prime} dX + P_t^N dt + \frac{1}{2} dX' P_{XX'}^N dX + dX' P_{Xt}^N dt + \frac{1}{2} dt P_{tt}^N dt \\ &= P_X^{N\prime} \left(\mu dt - KX_t dt + \Sigma_X' dZ_t\right) + P_t^N dt + \frac{1}{2} (dZ_t) \Sigma_X P_{XX'}^N \Sigma_X' dZ_t \end{aligned} \tag{16}$$

because in the limit dt tends to 0, the dt^2 and $dtdZ$ terms disappear and the dZ^2 term tends to dt. Substitution of this equation for the price changes and the nominal stochastic discount factor (6) into the fundamental valuation equation (15) brings about

$$0 = P_X^{N\prime} (\mu - KX_t) + P_t^N + \frac{1}{2} tr \left(\Sigma_X P_{XX'}^N \Sigma_X'\right) - P^N R_t - P_X^{N\prime} \Sigma_X' \Lambda_t \tag{17}$$

Note, the trace term (see Cochrane (2005), page 378) appears because only quadratic terms remain due to independence of the error terms. This partial differential equation has a solution of the form

$$P^N(X_t,t,t+\tau) = \exp\left(A^N(\tau) + B^N(\tau)' X_t\right) \tag{18}$$

In case of a single pay-off at time T, duration is defined as $\tau = T - t$. Substitute the derivatives

$$\begin{aligned} \frac{1}{P^N} P_X^N &= B^N \\ \frac{1}{P^N} P_t^N &= -\frac{1}{P^N} P_\tau^N = -\dot{A}^N - \dot{B}^{N\prime} X_t \\ \frac{1}{P^N} P_{XX'}^N &= B^N B^{N\prime} \end{aligned} \tag{19}$$

into the partial differential equation (17)

$$0 = B^{N\prime}(\mu - KX_t) + \left(-\dot{A}^N - \dot{B}^{N\prime} X_t\right) + \frac{1}{2} tr\left(\Sigma_X B^N B^{N\prime} \Sigma_X'\right) - R_0 - R_1' X_t - B^{N\prime} \Sigma_X' \left(\Lambda_0 + \Lambda_1 X_t\right) \tag{20}$$

to obtain explicit expressions for A^N and B^N . Note:
$tr\left(\Sigma_X B^N B^{N\prime} \Sigma_X'\right) = tr\left(B^{N\prime} \Sigma_X' \Sigma_X B^N\right) = B^{N\prime} \Sigma_X' \Sigma_X B^N$ because $tr(AB) = tr(BA)$. Both the stochastic term and the non stochastic term have to be zero, leading to

$$\dot{A}^N(\tau) = -R_0 - \left(\Lambda_0' \Sigma_X - \mu'\right) B^N(\tau) + \frac{1}{2} B^{N\prime}(\tau) \Sigma_X' \Sigma_X B^N(\tau) \tag{21}$$

$$\dot{B}^N(\tau) = -R_1 - \left(K' + \Lambda_1' \Sigma_X\right) B^N(\tau) \tag{22}$$

The nominal zero coupon bond with duration $\tau = 0$ and payout 1 has a price $P^N(X_t,t,t) = 1$, which implies $A^N(0) = 0$ and $B^N(0) = 0$. The instantaneous (*i.e.* given the state of the economy) nominal yield of a bond with duration zero (cash) is defined as $-d\ln P^N(X_t,t,t) = -\left(\dot{A}^N(0) + \dot{B}^N(0)' X_t\right) = R_0 + R_1' X_t \equiv R_t$. The instantaneous nominal yield of a bond with duration τ is $-d\ln P^N(X_t,t,t+\tau) = -\left(\dot{A}^N(\tau) + \dot{B}^N(\tau)' X_t\right)$. The differential equations can be solved in closed form

$$B^N(\tau) = \left(K' + \Lambda_1' \Sigma_X\right)^{-1} \left[\exp\left(-\left(K' + \Lambda_1' \Sigma_X\right)\tau\right) - I_{2\times 2}\right] R_1 \tag{23}$$

$$A^N(\tau) = \int_0^\tau \dot{A}^N(s) ds \tag{24}$$

with $I_{2\times 2}$ the two by two identity matrix. These relations will be used for market conform discounting of nominal liabilities of pension funds.

3.2 The real term structure

The fundamental pricing equation of a real zero coupon bond (thus a single payout at a fixed point in the future) is

$$E d\phi^R P^R = 0 \tag{25}$$

leading to the partial differential equation

$$0 = P_X^{R\prime}(\mu - KX_t) + P_t^R + \frac{1}{2} tr\left(\Sigma_X P_{XX'}^R \Sigma_X'\right) - P^R r_t - P_X^{R\prime}\Sigma_X'(\Lambda_t - \sigma_\Pi) \tag{26}$$

This partial differential equation has a solution of the form

$$P^R(X_t, t, t+\tau) = \exp\left(A^R(\tau) + B^R(\tau)' X_t\right) \tag{27}$$

in case of a single pay-off at time time $t+\tau$. Substitute the derivatives into the fundamental pricing equations leads to

$$0 = B^{R\prime}(\mu - KX_t) + \left(-\dot{A}^R - \dot{B}^{R\prime}X_t\right) + \frac{1}{2}B^{R\prime}\Sigma_X'\Sigma_X B^R - \left(\delta_{0r} + \delta_{1r}'X_t\right) - B^{R\prime}\Sigma_X'(\Lambda_0 - \sigma_\Pi + \Lambda_1 X_t) \tag{28}$$

Both the stochastic term and the non stochastic term have to be zero leading to

$$\dot{A}^R = -\delta_{0r} - \left[\left(\Lambda_0' - \sigma_\Pi'\right)\Sigma_X - \mu'\right] B^R + \frac{1}{2}B^{R\prime}\Sigma_X'\Sigma_X B^R \tag{29}$$

$$\dot{B}^R = -\delta_{1r} - \left(K' + \Lambda_1'\Sigma_X\right) B^R \tag{30}$$

The real zero coupon bond with duration $\tau = 0$ and payout 1 has a price $P^R(X_t, t, t) = 1$, which implies $A^R(0) = 0$ and $B^R(0) = 0$. The instantaneous real yield of cash is $-d\ln P^R(X_t, t, t) = -\left(\dot{A}^R(0) + \dot{B}^R(0)' X_t\right) = \delta_{0r} + \delta_{1r}' X_t \equiv r_t$ and of a bond with duration τ is $-d\ln P^R(X_t, t, t+\tau) = -\left(\dot{A}^R(\tau) + \dot{B}^R(\tau)' X_t\right)$. These relations will be used for market discounting of real liabilities of pension funds.

4 Bond funds implementing constant duration

The introduction of bond funds which implement constant duration is convenient to calculate the average return of a bond portfolio.This section follows Shi and Werker (2011) and Bajeux-Besnainou et al. (2003). Assume, a bond fund manager rebalances the portfolio permanently to hold the maturity τ constant, i.e. the fund invests only in bonds with maturity τ.

The development of the bond index of such a fund can be derived by applying the Itô-Doebling lemma to[3]

$$P^{F\tau}(X_t) = P^N(X_t, t, \tau) = \exp\left(A^N(\tau) + B^N(\tau)' X_t\right) \tag{31}$$

holding τ constant leads to

$$\begin{aligned} dP^{F\tau} &= P_X^{F\tau\prime} dX + P_t^{F\tau} dt + \frac{1}{2} dX' P_{XX'}^{F\tau} dX + dX' P_{Xt}^{F\tau} dt + \frac{1}{2} dt P_{tt}^{F\tau} dt \\ &= P^{F\tau} B^N(\tau)' dX + \frac{1}{2} P^{F\tau} dX' B^N B^{N\prime}(\tau) dX \end{aligned} \tag{32}$$

After substitution of the state equation (3) and using the Itô Doublin theorem brings about

$$\frac{dP^{F\tau}}{P^{F\tau}} = \left(B^N(\tau)' \mu - B^N(\tau)' K X_t + \frac{1}{2} B^{N\prime} \Sigma_X' \Sigma_X B^N\right) dt + B^N(\tau)' \Sigma_X' dZ_t \tag{33}$$

This equation together with stochastic discount factor (6) are consistent with the fundamental asset valuation equation if

$$E\left[\frac{dP^{F\tau}}{P^{F\tau}} + \frac{d\phi_t^N}{\phi_t^N} + \frac{dP^{F\tau}}{P^{F\tau}} \frac{d\phi_t^N}{\phi_t^N}\right] = 0 \tag{34}$$

which yields the restriction

$$B^N(\tau)' \mu - B^N(\tau)' K X_t + \frac{1}{2} B^{N\prime} \Sigma_X' \Sigma_X B^N - R_t - B^N(\tau)' \Sigma_X' \Lambda_t = 0 \tag{35}$$

Subsitution into equation (33) leads to the funds price dynamics equation

$$\frac{dP^{F\tau}}{P^{F\tau}} = \left(R_t + B^N(\tau)' \Sigma_X' \Lambda_t\right) dt + B^N(\tau)' \Sigma_X' dZ_t \tag{36}$$

Note $B^N(0) = 0$ leading to $\frac{dP^{F0}}{P^{F0}} = R_t dt$. These relations will be used to construct the portfolio return of pension funds.

5 Discretization of the differential equations

Both exact and numerical discretization of the model are used for estimation and simulation purposes. Both methods give the same results. However, exact discretization is numerically much more efficient. Both methods will be discussed subsequently.

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[3] Note, the fund's value index can not be determined using the instantaneous return of a bond with constant maturity. The instantaneous does not take into account changes in the state of the state.

5.1 Exact discretization

Exact discretization is possible by writing the whole model as a multivariate Ornstein-Uhlenbeck process

$$dY_t = (\Theta_0 + \Theta_1 Y_t)dt + \Sigma_Y dZ_t \tag{37}$$

with

$$Y' = \begin{bmatrix} X & \ln \Pi & \ln S & \ln P^{F0} & \ln P^{F\tau} \end{bmatrix}$$

in which X is the vector with the two state variables, Π the price index, S the stock index, P^{F0} the cash wealth index, $P^{F\tau}$ the bond wealth index with a duration τ, and Z the vector with the four independent Brownian motions extended with two zeros for cash and bond equations. Use Itô Doeblin theorem for log inflation

$$\begin{aligned} d\ln \Pi &= \frac{\partial \ln \Pi}{\partial \Pi} d\Pi + \frac{1}{2}\left(\frac{\partial^2 \ln \Pi}{\partial \Pi^2}\right)(d\Pi)^2 \\ &= (\pi_t dt + \sigma_{\Pi}' dZ_t) - \frac{1}{2}\left[\pi_t dt + \sigma_{\Pi}' dZ_t\right]^2 \\ &= (\pi_t - \frac{1}{2}\sigma_{\Pi}'\sigma_{\Pi})dt + \sigma_{\Pi}' dZ_t \end{aligned} \tag{38}$$

and log equity

$$\begin{aligned} d\ln S &= \frac{\partial \ln S}{\partial S} dS + \frac{1}{2}\left(\frac{\partial^2 \ln S}{\partial S^2}\right)(dS)^2 \\ &= (R_t + \eta_S)dt + \sigma_S' dZ_t - \frac{1}{2}\left[(r_t + \eta_S)dt + \sigma_S' dZ_t\right]^2 \\ &= (R_0 + R_1' X_t + \eta_S - \frac{1}{2}\sigma_S'\sigma_S)dt + \sigma_S' dZ_t \end{aligned} \tag{39}$$

Log wealth invested in a constant duration fund develops according to

$$\begin{aligned} d\ln P^{F\tau} &= \frac{\partial \ln P^{F\tau}}{\partial P^{F\tau}} dP^{F\tau} + \frac{1}{2}\left(\frac{\partial^2 \ln P^{F\tau}}{(\partial P^{F\tau})^2}\right)\left(dP^{F\tau}\right)^2 \\ &= \left(R_t + B^N(\tau)'\Sigma_X'\Lambda_t - \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N\right)dt + B^N(\tau)'\Sigma_X' dZ_t \end{aligned}$$

This implies for the multivariate Ornstein-Uhlenbeck process

$$d\begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} = \left(\begin{bmatrix} \mu \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ R_0 + \eta_S - \frac{1}{2}\sigma_S'\sigma_S \\ R_0 \\ R_0 + B^N(\tau)'\Sigma_X'\Lambda_0 - \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N \end{bmatrix} + \begin{bmatrix} -K & 0 \\ \delta_{1\pi}' & 0 \\ R_1' & 0 \\ R_1' & 0 \\ R_1' + B^N(\tau)'\Sigma_X'\Lambda_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ \ln\Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} \right) dt + \begin{bmatrix} \Sigma_X' \\ \sigma_\Pi' \\ \sigma_S' \\ 0 \\ B^N(\tau)'\Sigma_X' \end{bmatrix} dZ_t \tag{40}$$

After using the eigenvalue decomposition

$$\Theta_1 = UDU^{-1} \tag{41}$$

the exact discretization reads as

$$Y_{t+h} = \mu^{(h)} + \Gamma^{(h)}Y_t + \varepsilon_{t+h} \quad \text{and} \quad \varepsilon_{t+h} \sim N(0, \Sigma^{(h)}) \tag{42}$$

in which:

($\imath$.) $\Gamma^{(h)}$ is defined as

$$\Gamma^{(h)} = \exp(\Theta_1 h) = U\exp(Dh)U^{-1} \tag{43}$$

The matrix exponential is defined as

$$\exp(A) = I + \sum_{r=1}^{\infty}\frac{1}{r!}A^r \tag{44}$$

($\imath\imath$.) $\mu^{(h)}$ is defined as

$$\mu^{(h)} = UFU^{-1}\Theta_0 \tag{45}$$

where F a diagonal matrix with elements

$$F_{ii} = h\alpha(D_{ii}h) \tag{46}$$

with

$$\alpha(x) = \frac{\exp(x) - 1}{x} \quad \text{and} \quad \alpha(0) = 1 \tag{47}$$

($\imath\imath\imath$.) $\Sigma^{(h)}$ is defined as

$$\Sigma^{(h)} = UVU' \tag{48}$$

with

$$V_{ij} = \left[U^{-1}\Sigma_Y\Sigma_Y'(U^{-1})'\right]_{ij} h\alpha\left([D_{ii}+D_{jj}]h\right) \tag{49}$$

These relations are taken from Koijen et al. (2005) and Bergstrom (1984).

5.2 Numerical discretization

The Euler-Murayama approximation of equation (37) reads as:

$$Y_{t+h} = Y_t + \Theta_0 h + \Theta_1 Y_t h + h^{\frac{1}{2}}\Sigma_Y \Delta Z_t \tag{50}$$

The discrete time intervals are split up into smaller intervals to approximate continuous time. This very simple alternative did give the same results for small h. More precisely, using $h = 0.001$ in the Euler-Murayama approximation did give after 1000 steps the same results on a yearly basis as using $h = 1$ and one step in case of exact discretization. So, exact discretization is numerically much more efficient.

6 Risk neutral simulation

Equation set (40) together with stochastic discount factor (6) are consistent with the fundamental asset valuation equations, *i.e.* the expected discounted value of the price of an asset does not change over time. However, to value derivative assets exact knowledge of the risk exposure, *i.e.* the replicating portfolio, will be necessary. The determination of this replicating portfolio is very cumbersome and often even impossible. However, the system of equations[4]

$$d\begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} = \left(\begin{bmatrix} (\mu - \Sigma_X' \Lambda_0) \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi' \sigma_\Pi \\ R_0 - \frac{1}{2}\sigma_S' \sigma_S \\ R_0 \\ R_0 - \frac{1}{2} B^{N\prime} \Sigma_X' \Sigma_X B^N \end{bmatrix} + \begin{bmatrix} -(K + \Sigma_X' \Lambda_1) & 0 \\ \delta_{1\pi}' & 0 \\ R_1' & 0 \\ R_1' & 0 \\ R_1' & 0 \end{bmatrix} \begin{bmatrix} X \\ \ln \Pi \\ \ln S \\ \ln P^{F0} \\ \ln P^{F\tau} \end{bmatrix} \right) dt$$
$$+ \begin{bmatrix} \Sigma_X' \\ \sigma_\Pi' \\ \sigma_S' \\ 0 \\ B^N(\tau)' \Sigma_X' \end{bmatrix} d\widetilde{Z}_t \tag{51}$$

together with the stochastic discount factor[5]

$$\frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N} = -R_t dt \tag{52}$$

is consistent with the fundamental asset valuation equation, too. This makes the valuation of derivative products as net pension benefits easy because the portfolio composition of the pension fund becomes irrelevant for the determination of the discount factor in this risk-neutral setting. Indeed the discount factor is for all assets equal to R_t in the risk neutral setting. In summary: stochastic process (51) and discount factor (52) lead to the same expected value of the basic assets as (40) together with stochastic discount factor (6) starting from the values of the basic assets in a base year. But, stochastic process (51) and discount factor (52) are most convenient for the valuation of derivatives. Next subsections detail on the different equations of system (51).

6.1 Risk-neutral simulation and the term structure

The state equations (3) together with the stochastic discount factor (6) and fundamental pricing equation (12) bring about partial differential equation (17). Some reordering results in

$$0 = P_X^{N\prime} \left[\left(\mu - \Sigma_X' \Lambda_0\right) - \left(K + \Sigma_X' \Lambda_1\right) X_t \right] + P_t^N + \frac{1}{2} tr\left(\Sigma_X P_{XX'}^N \Sigma_X'\right) - P^N R_t \tag{53}$$

[4] The risk premium for equity, η_S, and bonds, $B^N(\tau)' \Sigma_X' \Lambda_t$, are not included in respectively the equity and bond fund equation of system (51), contrary to system (40). Moreover, the constant and slope coefficients of the state equations are different in both systems. The distribution of $\widetilde{Z}$ is again a Brownian motion, but it is another stochast than Z in system (40) in the sense that $\widetilde{Z}$ has another value than Z for every state of the world contrary to the values of $X, \Pi, \ldots$ which are the same in (40) and (51).

[5] Note the stochastic discount factor (52) in the risk neutral setting does not include the error terms as in equation (6).

The stochastic process for the state variables (the first row of equation 51)

$$dX_t = \left[\left(\mu - \Sigma_X' \Lambda_0 \right) - \left(K + \Sigma_X' \Lambda_1 \right) X_t \right] dt + \Sigma_X' dZ_t \tag{54}$$

together with the discount factor (52) and the fundamental valuation equation $Ed\widetilde{\phi}^N P^N = 0$ lead to partial differential equation (53), too, as can easily be proved. The term structure coefficients will be equal to (23) and (24) in this risk neutral setting by deduction. Equation (54) is part of system (51).

6.2 Risk neutral simulation equity

Equity equation (5) together with the stochastic discount factor (6) satisfies the fundamental asset equation (10) because restriction (11) is imposed. Dynamic equity equation

$$\frac{dS_t}{S_t} = R_t dt + \sigma_S' dZ_t \tag{55}$$

and stochastic discount (52) are consistent with the fundamental valuation equation $Ed\widetilde{\phi}^N S = 0$. The log-linear version of equation (55) is part of system (51).

6.3 Risk neutral simulation of bond funds implementing constant duration

Applying the Itô-Doebling lemma to

$$P^{F\tau}(X_t, t, \tau) = \exp\left(A^N(\tau) + B^N(\tau)' X_t \right) \tag{56}$$

holding τ constant leads to

$$\begin{aligned} dP^{F\tau} &= P_X^{F\tau\prime} dX + P_t^{F\tau} dt + \frac{1}{2} dX' P_{XX'}^{F\tau} dX + dX' P_{Xt}^{F\tau} dt + \frac{1}{2} dt P_{tt}^{F\tau} dt \\ &= P^{F\tau} B^N(\tau)' \left(\left[\left(\mu - \Sigma_X' \Lambda_0 \right) - \left(K + \Sigma_X' \Lambda_1 \right) X_t \right] dt + \Sigma_X' dZ_t \right) + \frac{1}{2} P^{F\tau} tr \left(\Sigma_X B^N B^{N\prime}(\tau) \Sigma_X' \right) dt \end{aligned} \tag{57}$$

using equation (54). After reordering follows

$$\frac{dP^{F\tau}}{P^{F\tau}} = \left(B^N(\tau)' \left(\mu - \Sigma_X' \Lambda_0 \right) - B^N(\tau)' \left(K + \Sigma_X' \Lambda_1 \right) X_t + \frac{1}{2} B^{N\prime} \Sigma_X' \Sigma_X B^N \right) dt + B^N(\tau)' \Sigma_X' dZ_t \tag{58}$$

This equation together with the stochastic discount factor (52) are consistent with the fundamental asset valuation equation $Ed\widetilde{\phi}^N P^{F\tau} = 0$ if

$$E\left[\frac{dP^{F\tau}}{P^{F\tau}} + \frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N} + \frac{dP^{F\tau}}{P^{F\tau}} \frac{d\widetilde{\phi}_t^N}{\widetilde{\phi}_t^N} \right] = 0 \tag{59}$$

or

$$B^N(\tau)'\mu - B^N(\tau)'KX_t + \frac{1}{2}B^{N\prime}\Sigma_X'\Sigma_X B^N - R_t - B^N(\tau)'\Sigma_X'\Lambda_t = 0 \tag{60}$$

This leads to the conclusion that the funds dynamic equation

$$\frac{dP^{F\tau}}{P^{F\tau}} = R_t dt + B^N(\tau)'\Sigma_X' dZ_t \tag{61}$$

together with the stochastic discount factor (52) lead to the same valuation of assets. The log-linear version of equation (61) is part of system (51).

7 Data

Figure 1 Nominal bond yields duration three months (Euribor) (left) and one year (right)

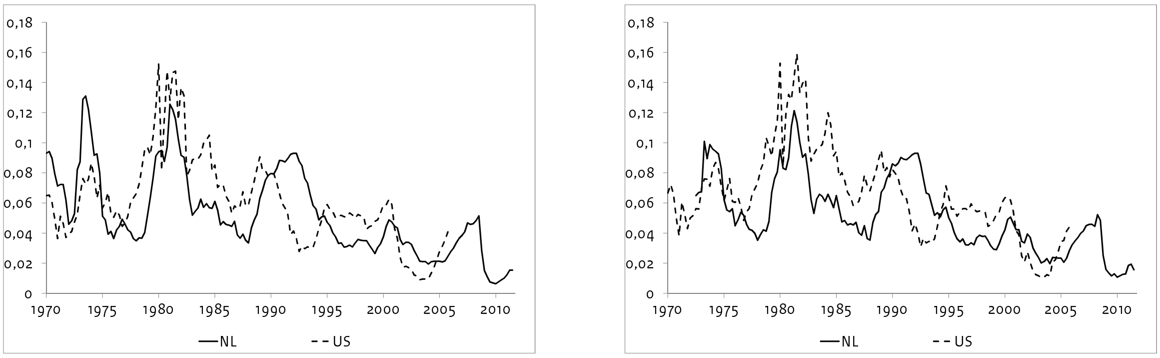
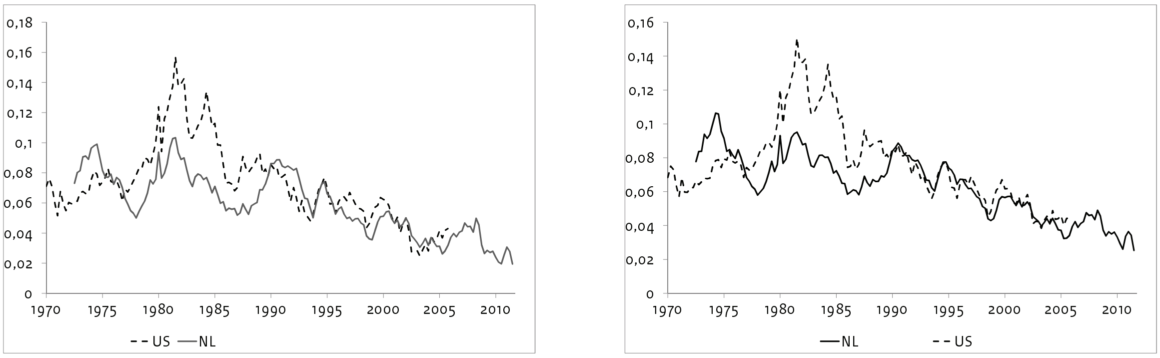
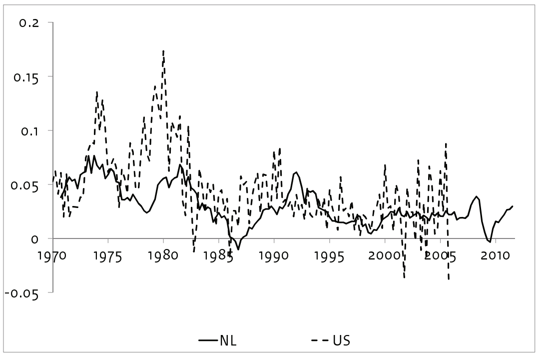


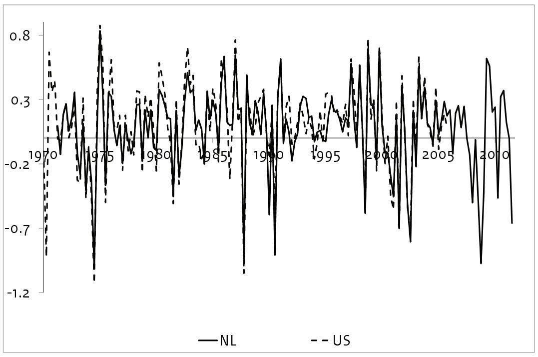
Figure 2 Nominal bond yields duration five year (left) and 10 year (right)



The data for the Netherlands are taken from Goorbergh et al. (2011).

Figure 3 Inflation (left) and equity return (right)





- Inflation: From 1999 on, the Harmonized Index of Consumer Prices for the euro area from the European Central Bank data website (http://sdw.ecb.europa.eu) is used. Before then, German (Western German until 1990) consumer price index figures published by the International Financial Statistics of the International Monetary Fund are included.
- Yields: Six yields are used in estimation: three-month, one-year, two-year, three-year, five-year, and ten-year maturities, respectively. Three-month money market rates are taken from the Bundesbank (www.bundesbank.de). For the period 1973:I to 1990:II, end-of-quarter money market rates reported by Frankfurt banks are taken, whereas thereafter three-month Frankfurt Interbank Offered Rates are included. Long nominal yields: From 1987:IV on, zero-coupon rates are constructed from swap rates published by De Nederlandsche Bank (www.dnb.nl). For the period 1973:I to 1987:III, zerocoupon yields with maturities of one to 15 years (from the Bundesbank website) based on government bonds were used as well (15-year rates start in June 1986). No adjustments were made to correct for possible differences in the credit risk of swaps, on the one hand, and German bonds, on the other. The biggest difference in yield between the two term structures (for the two-year yield) in 1987:IV was only 12 basis points.
- Stock returns: MSCI index from Fact Set. Returns are in euros (Deutschmark before 1999) and hedged for US dollar exposure.

The data for the US are taken from Koijen et al. (2010).

- Inflation: Data on the price index have been obtained from the Bureau of Labor Statistics. We use the CPI-U index to represent the relevant price index for the investor. The CPI-U index represents the buying habits of the residents of urban and metropolitan areas in the United States.
- Yields: Six yields are used in estimation: three-month, six-month, one-year, two-year,

Figure 4 Nominal yields US duration 3 month, 1, 5 and 10 years

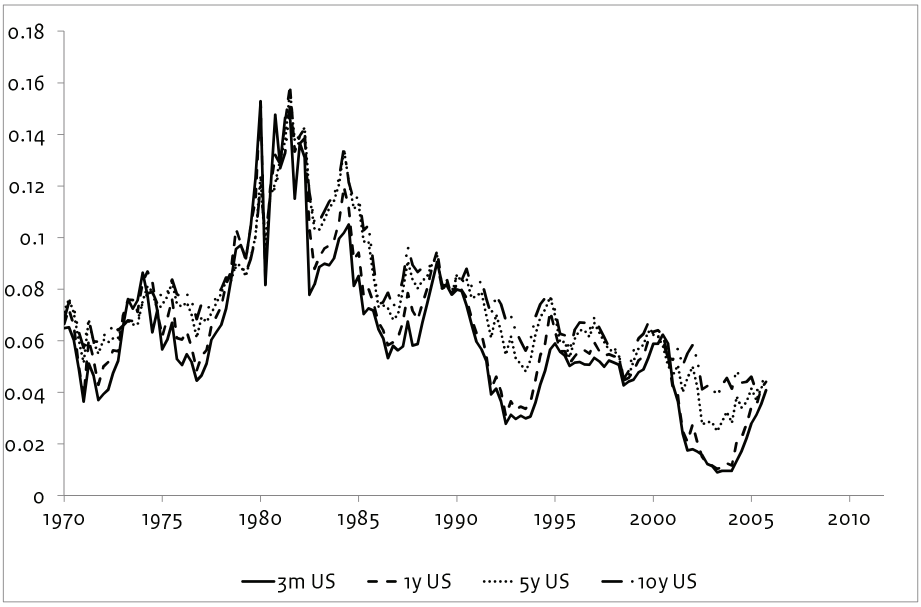
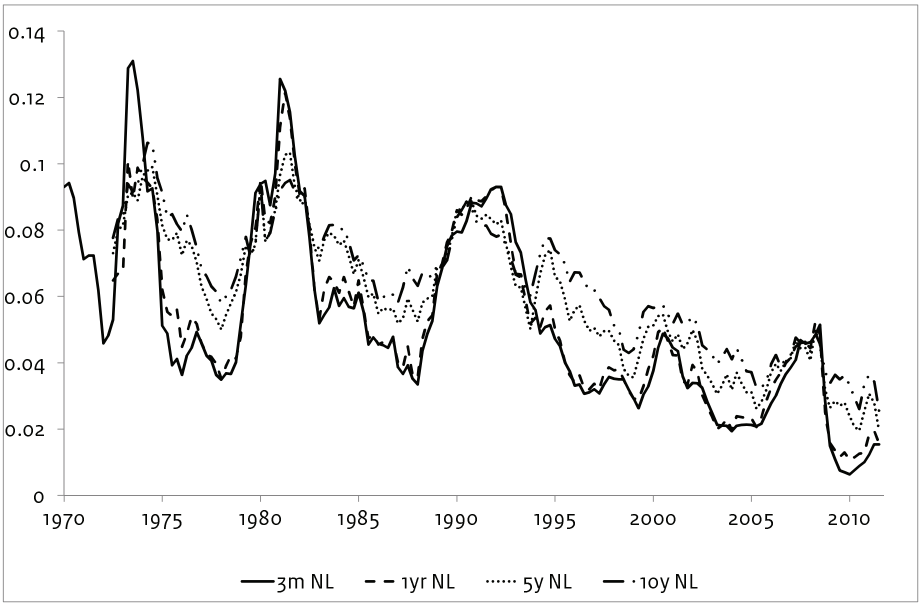


Figure 5 Nominal yields NL duration 3 month, 1, 5 and 10 years



five-year, and ten-year maturities, respectively. For the period January 1959-December 1998 the U.S. Government yield data are those of Duffee (2002) and Sangvinatsos and Wachter (2005). These data are taken from McCulloch and Kwon up to February 1991 and extended using the data in Bliss (1997) to December 1998. Koijen et al. (2010) extend the time series of one-year, two-year, five-year, and ten-year yields to December 2005 using data from the Federal Reserve Bank of New York. The data on the three-month and six-month yield are extended to December 2005 using data from the Federal Reserve Bank of St. Louis.

- Stock returns: Koijen et al. (2010) use returns on the CRSP value-weighted NYSE/Amex/Nasdaq index data for stock returns.

Comments:

- Decreasing bond rates over time after 1980
- Closer link between the short- and long-term yields in the US than in the Netherlands
- In the period 1973-1974, 1980-1981 and 1990-1991 inverse term structure in the Netherlands; In the year 1973 and 1980 in the US too, but less pronounced.
- Inflation US quarterly figures (multiplied by 4 to make them comparable with yearly data); inflation for the Netherlands year to year figures.
- Stock returns seem rather the same.

8 Estimation procedure

Assume, two yields are observed without measurement error. For those yields hold

$$y_t^\tau = \left(-A(\tau) - B(\tau)' X_t\right)/\tau$$

These observations can be used to determine the state vector X, given a set parameters which determine A and B. The other four yields are observed with a measurement error by assumption.

$$y_t^\tau = \left(-A(\tau) - B(\tau)' X_t\right)/\tau + \varepsilon_t^\tau \quad \text{and} \quad \varepsilon_t^\tau \sim N(0, \sigma^\tau)$$

Assume no correlation between the measurement errors. This system of measurement equations is extended with the equation set (42) for the state, inflation and equity, which we repeat for convenience

$$Y_{t+h} = \mu^{(h)} + \Gamma^{(h)} Y_t + \varepsilon_{t+h} \quad \text{and} \quad \varepsilon_{t+h} \sim N(0, \Sigma^{(h)}) \tag{62}$$

The likelihood is maximized with respect to the parameters using the method of simulated annealing of Goffe et al. (1994) to find the global optimum.

9 Estimation results

The estimation results for the US and the Netherlands are presented in Table 1. The significance of the estimates for the Netherlands is lower in general.

The unconditional expected inflation is in the US (4.2%) larger than in the Netherlands (2.2%). This explains the 2% larger unconditional nominal interest rate in the US. Both the instantaneous short rate and expected inflation are increasing in both $X1$ and $X2$. $X2$ is more persistent than $X1$. Moreover, the persistence is larger in the Netherlands than in the US for both variables, which explains partly the less significant parameter estimates. The first-order autocorrelation on an annual frequency equals 0.503 and 0.861 for the US and 0.725 and 0.906 for the Netherlands, respectively. The equity risk premium (η_S) is 5.4% for the US and 3.5% for the Netherlands. The unconditional price of risk of $X1$ is more negative than the price of risk of X2; i.e.,$|\Lambda_{0(1)}| > |\Lambda_{0(2)}|$. Table 2 reports the risk premium on nominal bonds along with their volatilities. The risk premium for bonds with a maturity of 10 years is 65 basis points higher in the Netherlands than in the US.

Table 1 Estimation results for the US and the Netherlands

	US		Netherlands	
Parameter	Estimate	(Standard error)	Estimate	(Standard error)
Expected inflation $\pi_t = \delta_{0\pi} + \delta_{1\pi}' X_t$				
$\delta_{0\pi}$	4.20%	(1.03%)	2.24%	(1.45%)
$\delta_{1\pi(1)}$	1.69%	(0.19%)	0.49%	(0.27%)
$\delta_{1\pi(2)}$	0.50%	(0.24%)	0.49%	(0.24%)
Nominal interest rate $R_0 + R_1' X_t$				
R_0	5.89%	(1.66%)	3.70%	(2.77%)
$R_{1(1)}$	1.92%	(0.14%)	1.40%	(0.43%)
$R_{1(2)}$	1.03%	(0.26%)	0.82%	(0.68%)
Process real interest rate and expected inflation $dX_t = \mu dt - KX_t dt + \Sigma_X' dZ_t$				
κ_{11}	0.687	(0.177)	0.32	(0.23)
κ_{22}	0.172	(0.035)	0.13	(0.13)
κ_{21}	−0.350	(0.119)	−0.23	(0.16)
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi' dZ_t$				
$\sigma_{\Pi(1)}$	0.02%	(0.05%)	−0.01%	(0.07%)
$\sigma_{\Pi(2)}$	0.11%	(0.04%)	−0.01%	(0.06%)
$\sigma_{\Pi(3)}$	0.98%	(0.03%)	0.60%	(0.04%)
Stock return process $\frac{dS_t}{S_t} = (r_t + \eta_S)dt + \sigma_S' dZ_t$				
η_S	5.38%	(2.48%)	3.52%	(3.88%)
$\sigma_{S(1)}$	−1.98%	(0.57%)	−0.16%	(1.71%)
$\sigma_{S(2)}$	−1.79%	(0.70%)	1.01%	(1.61%)
$\sigma_{S(3)}$	−1.74%	(0.68%)	−2.65%	(1.56%)
$\sigma_{S(4)}$	14.82%	(0.32%)	16.71%	(0.98%)
Prices of risk $\Lambda_t = \Lambda_0 + \Lambda_1' X_t$				
$\Lambda_{0(1)}$	−0.293	(0.127)	−0.271	(0.266)
$\Lambda_{0(2)}$	−0.158	(0.071)	−0.279	0.238)
$\Lambda_{1(1,1)}$	−0.103	(0.182)	0.167	(0.252)
$\Lambda_{1(1,2)}$	−0.101	(0.038)	−0.114	(0.239)
$\Lambda_{1(2,1)}$	0.503	(0.078)	0.395	(0.246)
$\Lambda_{1(2,2)}$	−0.168	(0.132)	−0.126	(0.140)

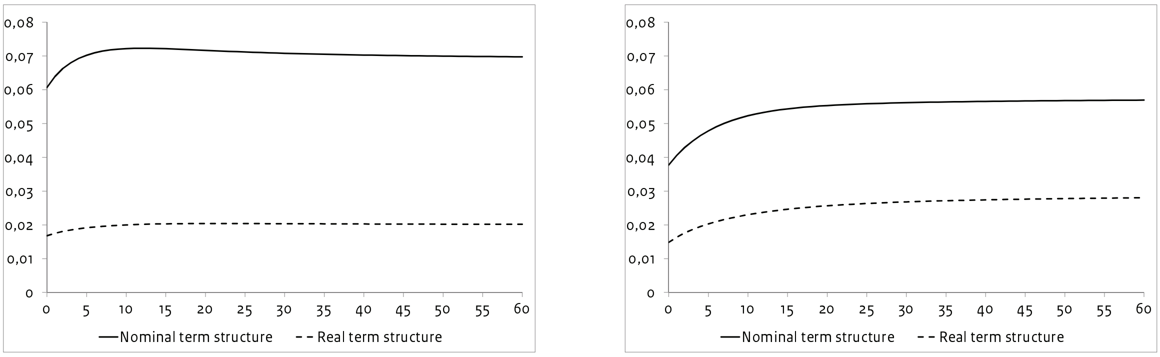
Table 2 Risk premia and volatilities

	US		Netherlands	
Maturities	risk premium	volatility	risk premium	volatility
One-year	0.58%	1.77%	0.53%	1.37%
Five-year	1.44%	6.36%	1.80%	5.1%
Ten-year	2.06%	11.75%	2.71%	9.36%

10 Simulations with estimated parameters

The simulations in this section give an impression of the dynamic characteristics of the model for the US and the Netherlands. The level of the nominal term structure (Figure 6) is in the US higher than in the Netherlands, contrary to the real term structure. This can be explained by the inflation differences. Both the unconditional expected value as well as the volatility of inflation is in the Netherlands lower (Figure 7). Although the unconditional nominal interest rate is lower in the Netherlands, the range of the interest rate seems rather the same due to the larger persistency in the Netherlands (Figure 8). The real rate and its volatility are both larger in the Netherlands (Figure 9).

Figure 6 Term structure US (left) and NL (right)



11 Conclusion

This paper estimates the capital market model of Koijen et al. (2010) using Dutch data. The link between the bond yields of different duration is not as large in north Europe as in the US. Moreover, the duration of the inverse term structures is in Europe longer than in the US. This may explain the less significant coefficient estimates for Europe. This is reflected by the

Figure 7 Inflation year on year US (left) and NL (right)

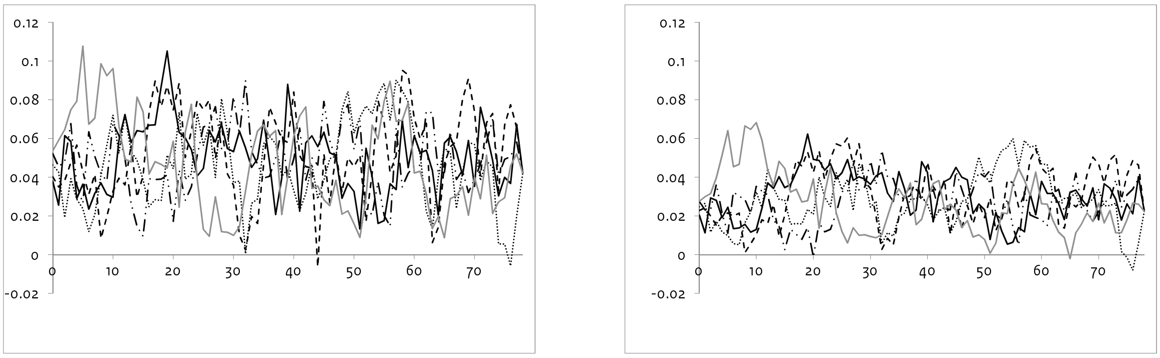


Figure 8 Nominal rate US (left) and NL (right)

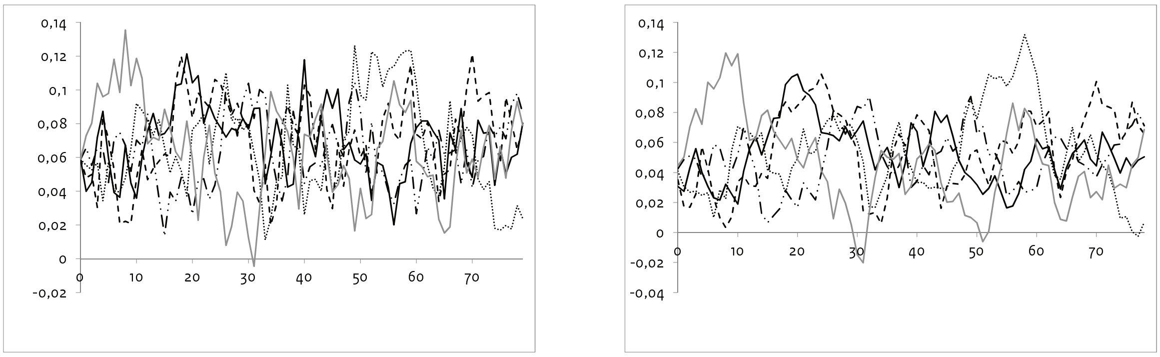
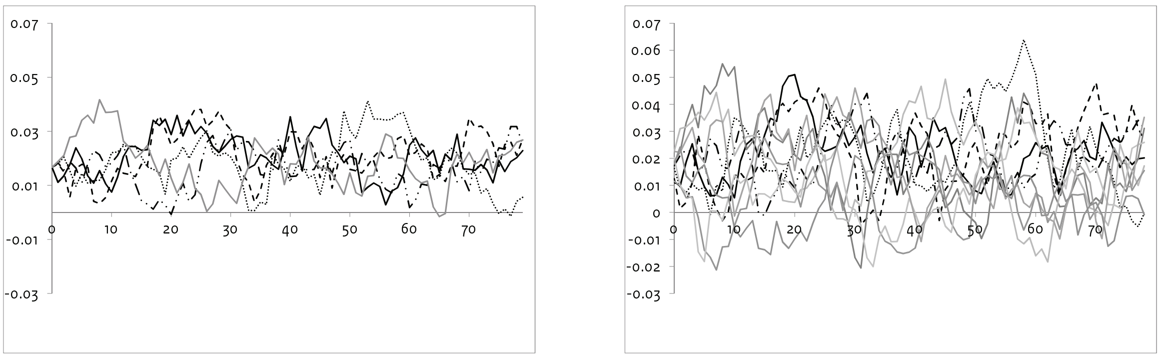


Figure 9 Real rate US (left) and NL (right)



significance of the estimation results. A possible explanation may be that the European capital market is not as closed as that of the US.

This parameter uncertainty has an important implication for the evaluation of the pension agreement. We need to check the robustness of the results by using alternative parameters.

References

Bajeux-Besnainou, I., J. V. Jordan, and R. Portait, 2003, Dynamic asset allocation for stocks, bonds, and cash, *The Journal of Business*, 76(2):263–288.

Bergstrom, A. R., 1984, *Handbook of Econometrics*, chapter Continuous Time Stochastic Models and Issues of Aggregation over Time, 1145–212, Elsevier.

Brennan, M. J. and Y. Xia, 2002, Dynamic asset allocation under inflation, *The Journal of Finance*, 3:1201–1238.

Broer, D. P., 2010, Macroeconomic risks and pension returns, CPB Memorandum 241, CPB.

Campbell, J. Y. and L. M. Viceira, 2001, Who should buy long-term bonds?, *The American Economic Review*, 91(1):99–127.

Cochrane, J. H., 2005, *Asset pricing*, Princeton university press.

Goffe, W. L., G. D. Ferrier, and J. Rogers, 1994, Global optimization of statistical functions with simulated annealing, *Journal of Econometrics*, 60(1-2):65–99.

Goorbergh, R. v. d., R. Molenaar, O. Steenbeek, and P. Vlaar, 2011, Risk models with jumps and time-varying second moments, Technical report, APG.

Hull, J. C., 2003, *Options, Futures and Other Derivatives*, Prentice Hall.

Koijen, R. S., T. E. Nijman, and B. J. Werker, 2005, Labor income and the demand for long-term bonds, Technical report, Center, Tilburg.

Koijen, R. S., T. E. Nijman, and B. J. Werker, 2006, Optimal portfolio choice with annuitization, Technical report, Center, Tilburg.

Koijen, R. S., T. E. Nijman, and B. J. Werker, 2010, When can life cycle investors benefit from time-varying bond risk premia?, *Review of Financial Studies*, 23(2):741–780.

Merton, 1992, *Continuous Time Finance*, Blackwell Publishing.

Sangvinatsos, A. and J. A. Wachter, 2005, Does the failure of the expectations hypothesis matter for long-term investors?, *The Journal of Finance*, 60(1):179–230.

Shi, Z. and B. J. Werker, 2011, Short horizon regulation for long term investors, Technical report, Finance and Econometrics Groups Tilburg University (forthcoming).

Shreve, S. E., 2004, *Stochastic Calculus for Finance I and II*, Springer.