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Abstract

We study incentives of financial intermediaries to reserve liquidity given that they can rely on the interbank market for their liquidity needs. Intermediaries can partially pledge their assets to each other, but not to the rest of the economy. Therefore liquidity provision is endogenous. We show that if the probability of a crisis is large or if assets are slightly pledgeable, then all intermediaries reserve liquidity. However, if the probability of a crisis is small or if assets are highly pledgeable, then intermediaries segregate ex ante: some reserve no liquidity, others reserve to the maximum and become liquidity providers. This segregation arises, because in the latter case the crisis short-term rate exceeds the returns on long-term investments, while at the same time higher liquidity holdings also increase survival probability. Together, these two effects result in increasing marginal returns to liquidity in the crisis state, and, consequently, segregation ex ante. In either equilibrium, aggregate liquidity is too small if assets are not fully pledgeable. Minimum liquidity requirements only improve welfare in the symmetric equilibrium. Marginally lowering the interest rate causes a marginal crowding-out of private liquidity with public liquidity in the symmetric equilibrium, but a full crowding-out in the asymmetric equilibrium.

Key Words: inside liquidity, partial pledgeability, interbank markets, minimum liquidity requirements, central bank.

JEL Classification: E43, G20, G33.

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1 Introduction

Financial intermediaries—banks, various funds, etc.—are among the primary providers of funding liquidity in the modern economy. However, intermediaries' own liquidity is subject to shocks: interbank loans may fail to roll over, firms may draw on credit lines, customers may unexpectedly withdraw deposits, or investors may redeem their shares. In extreme cases, such as the recent financial crisis of 2007–2009, liquidity shocks might trigger a banking crisis, thus causing a recession in the real economy.[1] To withstand liquidity shocks, intermediaries can either self-insure, by holding liquid assets to meet unexpected liquidity needs, or rely on borrowing liquid assets from other investors in the market for liquidity.

As has been emphasized in the literature on fire sales (originating from Shleifer and Vishny, 1992; Allen and Gale, 1994; Shleifer and Vishny, 1997), the group of outside investors, and their supply of liquidity in times of distress, may be limited: profitable lending to insiders requires expertise that is not immediately available to outsiders.

As a direct consequence of the lower pledgeability to outsiders, the amount of liquidity held by intermediaries for self-insurance purposes determines the aggregate liquidity available in the market. In times of large (aggregate) liquidity shocks, the price of liquidity will therefore rise rapidly, leading to high short-term interest rates. The high returns to liquidity in times of aggregate distress in turn give incentives to investment in liquidity ex ante. It is well known (see, e.g., Allen and Gale, 2004b) that in the absence of market failures, the investment incentives generated by such periods of high interest rates create optimal levels of liquidity.

When market failures are present, however, intermediaries may underinvest in liquidity, and regulation of liquidity (either ex ante or ex post) may be called for. The goal of this paper is to explore how investment in liquidity is affected by one particular market failure, namely imperfect pledgeability of future cash flows on the interbank market (as in Holmström and Tirole, 1998), and how regulation of liquidity can correct for this market failure.

We study intermediaries' ex ante choice to reserve liquidity given that in case of a liquidity shock they can rely on the interbank market for their liquidity needs. We find that for sufficiently infrequent liquidity shocks the equilibrium is asymmetric, where some intermediaries invest only in liquidity, while others focus on long-term investments. In contrast, if liquidity shocks occur relatively often the equilibrium is symmetric, and all intermediaries invest part of their assets in liquid reserves. Importantly, the effectiveness of liquidity policy depends on the type of equilibrium that prevails. In the symmetric equilibrium imposing minimum liquidity requirements or opting for central bank intervention alleviates the underprovision of liquid-

[1] For an overview of this crisis see, e.g., Brunnermeier (2009).

ity. In the asymmetric equilibrium minimum liquidity requirements are not effective while the central bank intervention causes a full crowding-out of private liquidity.

Our model has two types of assets: long-term investments and safe liquid investments (cash). In the spirit of Diamond and Dybvig (1983), there are three dates. On date one intermediaries spend a part of their initial cash for long-term investments, and keep the remaining cash as reserves. On date two with some probability there is a crisis and then each intermediary experience a uniformly distributed idiosyncratic liquidity shock on the liability side of its balance sheet. Intermediaries short of liquidity can pledge part of the cash flow from their long-term investments to borrow from the intermediaries with liquidity surpluses, at an endogenous interbank rate. We consider a competitive interbank market for liquidity, where each individual participant has negligible influence on the short-term interest rate. Distressed intermediaries that are unable to borrow sufficient liquidity go bankrupt. On date three the long-term investments of surviving intermediaries yield returns and the debt contracts are settled.

The intermediaries do not fully take into account that their combined liquidity reserves lower the crisis interest rate and ease the credit rationing faced by intermediaries in distress who would have positive net present value when saved. This externality results in underprovision of liquidity in laissez-faire markets. However, if investments are fully pledgeable to insiders, then there is no credit rationing and liquidity provision is optimal.

Given moderate values for pledgeability and the likelihood of a crisis, liquidity reserves exhibit decreasing marginal returns. Consequently, intermediaries choose to diversify their portfolios by holding both investments and liquidity reserves on their books. As we explain further, in this case if a crisis happens the interest rate is relatively low and we will call such a crisis a normal crisis (see Fig. 2).

If pledgeability is high or if the likelihood of a crisis is small, the situation changes. Intermediaries then keep low liquidity reserves resulting in a high interest rate when a crisis occurs. During a crisis the short-term interest rate may in fact spike above the returns on long-term investments. We call such a crisis severe. When a severe crisis hits, liquidity reserves exhibit increasing marginal returns: higher liquidity not only increases an intermediary's survival probability, as it does in the normal crisis, but it also yields higher returns than long-term investments. In this case, ex ante profits become convex in liquidity reserves, and intermediaries segregate: some bet on the absence of a crisis, and fully invest in long term assets; others bet on the occurrence of a crisis, and fully invest in liquid reserves.

Finally, in the extreme case when pledgeability is very low, intermediaries mostly rely on themselves for liquidity rather than on the interbank market. Consequently, the amount of inside liquidity in case of crisis is large, whereas the amount traded is small due to severe credit rationing (low pledgeability).

The crisis interest rate hits zero, and we refer to this situation as autarky.

The financial crisis of 2007–2009 triggered a range of policy responses. Ex-post, the Federal Reserve, the Bank of England, and the European Central Bank responded with lowering interest rates through open-market operations and through quantitative easing. In order to alleviate future crises the Basel Committee on Banking Supervision introduced an ex-ante regulation in Basel III: minimum liquidity requirements. We evaluate these policy responses in the context of our model.

In the normal crisis regime all intermediaries reserve liquidity. Introducing minimum liquidity requirements then pushes-up the liquidity reserves across all intermediaries, improving welfare. In the severe crisis regime, however, a fraction of intermediaries act as pure long-term investors. Minimum liquidity requirements not only directly decreases investments of those intermediaries but also their numbers, because mixing investments and liquidity reserves is less profitable than pure specialization. While aggregate liquidity reserves increase with the introduction of minimum liquidity requirements, this double loss of investments offsets the benefits. Consequently, minimum liquidity requirements decrease welfare in this regime.

As a second policy response we consider ex post intervention. If there is insufficient liquidity on date two, the central bank can issue new cash and act as an additional lender on the interbank market. Such policy has two potential consequences: inflation, and crowding out of private liquidity with public liquidity. Inflation lies outside the scope of our paper, and thus we do not address the question what the optimal interest rate should be. Instead we explore the marginal benefits of such an ex-post intervention. To do so, we consider a scenario in which the central bank commits to a marginal decrease in the short-term interest rate in case of a crisis, and we analyse the resulting crowding-out effect.

Suppose there is a risk of a normal crisis, and the central bank commits to lower the interest rate marginally if a crisis actually happens. In this case, liquidity reserves are characterized by decreasing marginal returns, and intermediaries diversify their capital between investments and liquidity reserves. Anticipating a lower interest rate in case of a crisis, the intermediaries decrease their liquidity reserves. For a marginal decrease in the interest rate, there will be a marginal crowding-out effect. Conversely, suppose that there is a risk of a severe crisis. In this case liquidity reserves are characterized by increasing marginal returns, and some intermediaries are pure long-term investors, while others hoard liquidity. A marginally lower interest rate in case of a crisis makes liquidity hoarding marginally less profitable than pure investing. Consequently, all intermediaries become pure investors. A commitment by the central bank to marginally lower the interest rate in case of a crisis causes full crowding out of private liquidity with public liquidity.

The financial crisis of 2007-2009 was deemed unlikely, and pledgeability of investments was improving in the fore-run to the crisis. Arguably, this crisis then belongs to the severe region in our classification. If we believe that markets expected the major central banks to keep the short-term interest rates low if a crisis were to occur, then this past financial crisis provides anecdotal support for our result of full crowding out of private liquidity with public liquidity.

The paper is organized as follows. The following Section 2 discusses the relevant literature. Section 3 sets up a formal model. Section 4 solves the model absent any regulation or intervention. Minimum liquidity requirements are discussed in Section 5, central bank policy is discussed in Section 6. Concluding remarks are sketched in Section 7.

2 Related literature

Our work relates to the literature on bank liquidity and maturity transformation, and to work on limits to arbitrage and fire-sale pricing in the presence of capital constrained specialist speculators. We add to this work the insight that intermediaries may endogenously segment into liquidity providers and long-term investors, and that effectiveness of policy measures is affected by this endogenous segmentation.

The basic framework that links liquidity and investments, and that was widely adopted by the literature, was proposed by Diamond and Dybvig (1983). There are three dates, intermediaries make long-term investments on date one, these investments yield returns on date three, and on date two intermediaries experience liquidity shocks due to consumers' withdrawals. These liquidity shocks have a real impact on the economy, because they cause a partial liquidation of the long-term investments. Bhattacharya and Gale (1987) extend this framework with an interbank market on date two. They show that if there is asymmetric information about liquidity shocks and if depositors are risk-averse, then the interbank market creates a free-rider problem and intermediaries underinvest in the liquid asset. This result suggests an argument for minimum liquidity requirements. Allen et al. (2010) employ a similar model but focus instead on the price volatility in the interbank market that is due to aggregate liquidity shocks. Assuming risk-averse depositors, the authors provide a formal support to the argument that one of the welfare-improving goals of a central bank is smoothing the short-term interest rate (see, e.g., Goodfriend and King, 1988). Both Bhattacharya and Gale (1987) and Allen et al. (2010) focus on withdrawals of deposits as source of liquidity shocks, i.e. they focus on liability side liquidity shocks.

We also focus on the role of intermediaries and analyse how an interbank market at date two allows intermediaries to meet a liability side shock. However, in our paper the source of market failure is limited pledgeability, as in Holmström and Tirole (1998, 2011). While these authors also follow the basic three-stage framework of Diamond and Dybvig, they consider risk-neutral firms, asset side liquidity shocks, and they focus on a single source of inefficiency, namely on a partial pledgeability of future income. Future income is partially pledgeable if only a part of it is liquid, i.e. if only a part can be readily traded or used as a collateral. Partial pledgeability is a simple case of credit rationing and as such can be explained using agency costs (Stiglitz and Weiss, 1981; Holmström and Tirole, 2011).

Holmström and Tirole (1998) show that if there are only idiosyncratic liquidity shocks on date two, and if there is centralized intermediation, then the free market can efficiently provide for liquidity; however, if there is a pure aggregate liquidity shock, then additional public provision of liquidity is required. A study related in method is Caballero and Krishnamurthy (2001), who apply the idea of partial pledgeability to international credit markets. The authors show that partial pledgeability of domestic assets creates a wedge between internal and external rates of return and, consequently, firms underinvest in international collateral.

In considering only a limited set of players who have the skills to supply liquidity on date two (the rival intermediaries) we follow the literature on fire-sales, or 'cash-in-the-market prices' (Shleifer and Vishny, 1992; Allen and Gale, 1994; Shleifer and Vishny, 1997).[2] That line of research focuses on endogenous prices of liquid assets after a liquidity shock. In our paper, it is not asset prices but rather the price for credit that is affected, however we share the approach of assuming inelastic supply of liquidity after the shock. While a large part of the fire-sales literature exogenously separates those who can provide liquidity from those who need it, in the present paper this distinction is endogenous. In this respect, our work is close to Acharya et al. (2009), who also study a setting where ex ante identical banks reserve liquidity, taking into account the prospective profits from purchasing assets at fire-sale prices from rivals hit by a shock.

Our paper's main novelty is in analysing the possibility of an endogenous segregation into liquidity providers and long-term investors, and the consequences this segregation has on policy. Though almost all papers focus on either symmetric equilibria, or on an exogenous group of liquid speculators, the result that the equilibrium need not be symmetric was already briefly pointed out in Allen and Gale (2004a), in a model of fire sales. We explore extensively when such equilibria occur as a function of the degree of market failure, and consider the implication of such asymmetric equilibria on policy.

Concurrently with our work, Carletti and Leonello (2011) obtain a similar result in a different setting. The authors focus on competition between banks in a setting with liquidity shocks, fire sales and risk-averse depositors. They show that if competition is mild, then each bank diversifies between liquid and illiquid assets, whereas if competition is strong, the banks seg-

[2] See Shleifer and Vishny (2011) for a recent survey.

regate: some hoard liquid assets and others fully invest in illiquid assets. That a result similar to ours also obtains in Carletti and Leonello's setting hints at its generality. Further, both our paper and theirs provide testable predictions as to when to expect segregation among financial intermediaries.

3 Model Setup

There are three dates and N financial intermediaries in the model. We think of these intermediaries as hedge funds, mutual funds, and other collective investment schemes, and we refer to them, for brevity, as funds. We are interested in the competitive limit: $N \to \infty$. On date one the funds divide their assets between long-term investments and cash. Investments are partially pledgeable to insiders (other funds, the central bank) and non-pledgeable to outsiders (the rest of the economy). Cash is fully pledgeable to both insiders and outsiders. There is a likelihood that a crisis occurs on date two. In the case of a crisis, funds experience various liquidity shocks on the liability side of their balance sheets. The funds with liquidity shortages can partially pledge their investments to borrow on an interbank market from the funds with liquidity surpluses. Additionally, the central bank can inject extra liquidity. The distressed funds that are unable to borrow sufficient liquidity go bankrupt. On date three the investments yield returns and the interbank debt contracts are settled.

This setup is purposefully stylized. A more realistic setup would explicitly account for the agency conflicts between funds and their original financiers, outside liquidity (when investments are also partially pledgeable to outsiders), market power (small N instead of our $N \to \infty$), and for the dynamic nature of the problem (we only consider a one-time shock). However, we abstract from these generalizations so as to study the basics of liquidity provision when interbank markets are characterized by partial pledgeability.

We now proceed with a detailed description of the model, which is summarized in Fig. 1. Before date one each fund 1 unit of cash. On date one a fund i, $i \in \{1, \ldots, N\}$, chooses to invest $1 - \ell_i$ into long-term projects and to keep the remaining ℓ_i as liquidity reserves.

On date two with probability q, $0 < q < 1$, there is a crisis. In the case of a crisis, a fund i experiences an idiosyncratic withdrawal of funds in amount of ε_i, where shocks ε are i.i.d. with $\varepsilon_i \sim U[0, 1]$. Generally speaking, these shocks are liability side liquidity shocks. With probability $1 - q$ there is no crisis, in which case $\varepsilon_i \equiv 0$.[3]

[3]Many institutions are financed with short-term funding, and we can interpret shocks ε_i as sudden withdrawals of such funds. These might reflect, e.g., withdrawals of deposits or equity, or an inability to roll-over previous debt, for instance as a result of a sudden stop in funding markets to which intermediaries are exposed (see e.g. Brunnermeier and Pedersen,

Figure 1: Timeline

	1. Funds make portfolio choices.	2. With probability q liquidity shocks ε_i occur. Surviving funds reallocate liquidity through the interbank market.	3. Long-term projects yield returns R, interbank loans are settled.
Cash	ℓ_i	0	$(1+R)(1-\ell_i)+$ $(1+r)(\ell_i-\varepsilon_i)$
Investments	$1-\ell_i$	$1-\ell_i$	0
Interbank loans	0	$\ell_i-\varepsilon_i$	0

Because investments are not pledgeable to outsiders, the funds have to finance the liquidity shocks either from their liquid reserves or through the interbank market. If a fund i has enough reserves to meet its obligations, i.e. if $\ell_i \geq \varepsilon_i$, then the fund lends the remaining cash $(\ell_i - \varepsilon_i)$ in the interbank market. Conversely, if a fund i has a shortage of cash, then it pledges its long-term assets to borrow the required amount $(\varepsilon_i - \ell_i)$ in the interbank market, given it has sufficient pledgeable assets. Otherwise, it goes bankrupt. We briefly postpone the discussion of how much assets funds can pledge. For a borrower in the interbank market, the position in interbank loans $\ell_i - \varepsilon_i$ in Fig. 1 is negative.

Funds with liquidity surpluses and those with liquidity shortages form, respectively, liquidity supply and liquidity demand on the interbank market. The interbank market is a Walrasian market: there is a clearing short-term interest rate r, at which all funds lend and borrow. The short-term rate r is therefore endogenous, and we solve for it in the following section. Additionally, the central bank can inject extra liquidity thus lowering the interest rate. We analyse the effects of the central bank's policy in Section 6.

On date three the long-term investments yield a gross cash return of $1 + R$ per unit invested, with $R > 0$. Further, the interbank debt contracts are settled. Surviving funds' total assets after date three then amount to

$$(1 + R)(1 - \ell_i) + (1 + r)(\ell_i - \varepsilon_i). \tag{1}$$

Funds survive the crisis of date two if their pledgeable assets are sufficient to secure a loan covering their liquidity deficit. If there are no agency and

2009). Other examples might be credit line obligations, that are drawn down, or payments that intermediaries have to make to support affiliated entities in distress. Formally, we abstract from the exact source of the liquidity shocks in our model and we treat these shocks as exogenous (as do other recent contributions in the fire-sale literature, such as for instance Diamond and Rajan, 2011; Krishnamurthy, 2009; Caballero and Simsek, 2011). Rigorous inclusion of those types of shocks will necessarily change the technical aspects of our model (e.g., we will have to account for the initial debt and solve for the contracts with depositors), but we do not expect our general results to change qualitatively, as should become clear from the following discussions of those results.

transaction costs, then a distressed fund i can get a loan on date two as long as its final equity is non-negative, i.e. as long as

$$(1+R)(1-\ell_i) \geq (1+r)(\varepsilon_i - \ell_i).$$

In practice, agency and transaction costs are always present. Consequently, the future returns from investments are only partially pledgeable (see Holmström and Tirole, 2011). Let θ denote the share of returns that is pledgeable, with $0 < \theta < 1$. Then a distressed fund i faces the following credit constraint on date two:

$$\theta(1+R)(1-\ell_i) \geq (1+r)(\varepsilon_i - \ell_i). \tag{2}$$

For the sake of clarity, Appendix A shows how agency costs of debt can be used to explain partial pledgeability and the resulting credit constraint.[4]

If a fund meets its credit constraint, then its final value after the crisis has passed is given by (1). If it does not meet its credit constraint, then it becomes insolvent. Consequently, the fund files for bankruptcy and its final value is zero. On date one the manager of every fund chooses $\ell_i \in [0,1]$ so as to maximize the expected final value.

In the following analysis we normalize the total assets among funds to one and, as we focus on competitive markets, we consider $N \to \infty$.

4 Laissez-faire Equilibria

We start the analysis of equilibrium liquidity holdings ℓ by searching for a symmetric equilibrium. We find that such an equilibrium exists in part of the parameter region. We denote this region as the *normal* region, and identify associated equilibrium quantities with a subscript n. With probability q, a crisis occurs, and all funds experience a uniform i.i.d. liquidity shock. In the crisis state, there will be interbank trade on date 2. The short-term interest rate r_n in this second stage will be such that it clears the interbank market for liquidity. With probability $1-q$ no shocks occur, and there is no need for trade in the interbank market.

Individual funds take the crisis short-term rate r_n as given, i.e., liquidity markets are perfectly competitive. On date 1, each fund i maximises its expected final date value by keeping a fraction ℓ_i of its assets liquid, and investing the remainder $1-\ell_i$ in illiquid assets yielding $1+R$ on the final date.

Fund i's expected final value is given by

$$\begin{aligned} v_i(\ell_i) =& (1-q)\left((1+R)(1-\ell_i)+\ell_i\right) \\ &+ q\int_0^{\bar{\varepsilon}(r_n,\ell_i)} \left((1+R)(1-\ell_i) + (1+r_n)\left(\ell_i - \varepsilon\right)\right) d\varepsilon. \end{aligned}$$

[4] For a discussion on agency costs of debt see, e.g., Jensen and Meckling (1976), Stiglitz and Weiss (1981).

The first term corresponds to the no-crisis state, in which all illiquid assets yield a rate $1+R$, and there is no trade in liquid assets. The second term involves the crisis state. A fund will only survive the crisis if faces a sufficiently low shock, $\varepsilon_i \leq \bar{\varepsilon}(r_n, \ell_i)$. If it experiences a shock ε_i lower than its liquid holdings ℓ_i, it will lend its liquidity surplus $\ell_i - \varepsilon_i$ in the interbank market, at the short-term rate r_n. If $\varepsilon_i > \ell_i$, the fund needs to borrow the deficit in order to avoid bankruptcy. The maximum shock a fund can sustain, $\bar{\varepsilon}(r_n, \ell_i)$, is defined by the requirement that it satisfies the credit constraint (2) with equality when borrowing in the interbank market,

$$\theta(1+R)(1-\ell_i) = (1+r_n)(\bar{\varepsilon}(r_n, \ell_i) - \ell_i).$$

Finally, the crisis short-term rate in the symmetric equilibrium, r_n is defined by market clearing in the interbank market:

$$\int_0^{\bar{\varepsilon}(r_n, \ell_n)} (\ell_n - \varepsilon) d\varepsilon = 0,$$

where ℓ_n denotes the equilibrium liquidity holdings. From this market clearing condition, we conclude that in equilibrium, $\bar{\varepsilon}(r_n, \ell_n) = 2\ell_n$.

Funds optimize their value v_i over ℓ_i, taking r_n as given. Solving the first-order condition and substituting for r_n using $\bar{\varepsilon}(r_n, \ell_n) = 2\ell_n$, to reflect market clearing, yields a candidate symmetric equilibrium with

$$\ell_n = \frac{1}{2} - \frac{1-\theta+\beta}{4},$$

where we have defined

$$\beta = \frac{(1-q)R}{q(1+R)}.$$

The short-term rate in the crisis state equals

$$1 + r_n = \theta(1+R)\frac{2+(1-\theta+\beta)}{2-(1-\theta+\beta)}.$$

We obtain that r_n is strictly positive as long as $\theta > \theta_1(q, R)$, where

$$\theta_1(q, R) = 1 - \frac{\sqrt{(1+2q)^2 R^2 + 8qR} - R}{2q(1+R)}.$$

To verify that the solution ℓ_n maximizes the fund's utility, we consider the second derivative in the optimum. For this to be negative, we need to impose

$$\theta < 1 - \beta,$$

in which case the fund's value is concave in ℓ_i.

We summarize this in

Proposition 1. *If $\theta_1 < \theta < 1-\beta$, then a unique equilibrium exists. This equilibrium is symmetric: each fund reserves a fraction of liquid assets in amount of ℓ_n, where*

$$\ell_n = \frac{1}{2} - \frac{1-\theta+\beta}{4}.$$

In the case of crisis the gross interest rate

$$1+r_n = \theta(1+R)\frac{2+(1-\theta+\beta)}{2-(1-\theta+\beta)}.$$

The normalized expected final value of any fund equals v_n, where

$$v_n = (1+R)q\left(\frac{1-q}{q} + \frac{1}{8}\left((2-\beta)^2-(1-\theta)^2\right)\right).$$

Formal proofs for this and following propositions are presented in the appendix.

Given some level of pledgeability θ, when β is large, or equivalently, the probability q of a crisis occurring is small, the symmetric solution is no longer an equilibrium, as the value function becomes convex in ℓ_i. The intuition is that, when the probability of crisis is small, the short-term rate r in the crisis state can become (much) larger than the long-term return on illiquid assets, R. In this case, a fund's pay-off in times of crisis will be increasing and convex in ℓ_i. Besides raising a fund's survival probability by increasing the maximum shock $\bar{\varepsilon}(r, \ell_i)$ that it can sustain, a greater share of liquid assets ℓ_i will also be more profitable, as the return on liquid assets r_n is greater than than the return on illiquid ones R. In the no-crisis state, the fund's pay-off is linear and decreasing in ℓ_i. Putting these two observations together, the fund's value will be convex, rather than concave, in ℓ_i, when r_n is sufficiently large.

Motivated by this observation, we search for an asymmetric equilibrium in case β is large (i.e. q small). Crises are less frequent, but liquidity shortage in case of a crisis is more severe. We call this region the *severe* region, and use subscript s to denote equilibrium quantities. In the asymmetric equilibrium, funds specialise: some (a fraction α) keep all their assets liquid, $\ell_i = 1$, while the others choose $\ell_i = 0$ and invest in long-term illiquid assets only. This can only be an equilibrium if expected pay-offs are equal in either case:

$$\begin{aligned} v(\ell_i = 0) &= (1-q)(1+R) + q\theta\frac{(1+R)^2}{1+r_s}\left(1-\frac{\theta}{2}\right) \\ &= 1-q+q\frac{1+r_s}{2} = v(\ell_i = 1), \end{aligned}$$

where we used (4) for the cases $\ell_i = 0, 1$, and substituted

$$\bar{\varepsilon}(r_s, \ell_i = 0) = \theta\frac{1+R}{1+r_s}, \quad \bar{\varepsilon}(r_s, \ell_i = 1) = 1. \tag{3}$$

This equality allows us to solve for the interest rate r_s in the severe region,

$$1 + r_s = (1 + R)\left(\beta + \sqrt{\beta^2 + 1 - (1-\theta)^2}\right). \tag{4}$$

Assuming market clearance, the fraction α of funds investing in liquid assets ($\ell_i = 1$) should be such that in the crisis state, liquidity demand (from funds with $\ell_i = 0$) equals liquidity supply (from those with $\ell_i = 1$) at the given interest rate,

$$\frac{1}{2}\alpha = \frac{(1-\alpha)}{2}\bar{\varepsilon}(r_s, \ell = 0)^2$$

or

$$\alpha = \frac{\theta^2(1+R)^2}{\theta^2(1+R)^2 + (1+r_s)^2}.$$

Again we have to verify that at this interest rate, a fund's profit function is indeed convex, so that no fund wants to deviate to an interior value of ℓ_i. We find that this holds precisely when

$$\theta > 1 - \beta.$$

Summarizing this analysis for the severe crisis region, we have

Proposition 2. *If $\theta > 1 - \beta$, then a unique equilibrium exists. This equilibrium is asymmetric: a share α of funds keep all their equity liquid, while the remaining $1 - \alpha$ funds reserve zero liquidity, where*

$$\alpha = \frac{\theta^2(1+R)^2}{\theta^2(1+R)^2 + (1+r_s)^2}.$$

In the case of crisis the gross interest rate

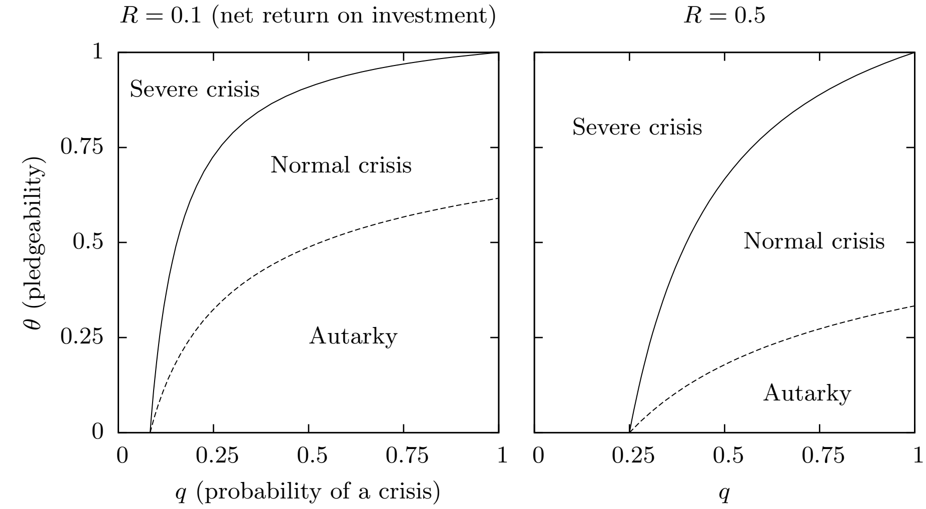
$$1 + r_s = (1 + R)\left(\beta + \sqrt{\beta^2 + 1 - (1-\theta)^2}\right).$$

The normalized expected final value of any fund equals v_s, where

$$v_s = \frac{q(1+r_s)}{2} + (1-q).$$

So far in the analysis, we assumed that liquidity supply equals liquidity demand in the crisis state: the interbank market clears at a positive interest rate r. This cannot continue to hold as pledgeability θ drops to zero: if long-term assets are not pledgeable at all, an interbank market cannot exist and we will be left with a situation of autarky. At $\theta = 0$, each fund will save some liquidity to protect itself against liquidity shocks. Funds with low shocks will survive the crisis with excess liquidity, while those with higher shocks will go bankrupt. For small positive θ, a qualitatively similar outcome prevails, where in the crisis state liquidity supply exceeds liquidity demand. In this region, which we refer to as the *autarky* region (subscript a), we find a symmetric equilibrium with $r_a = 0$:

Figure 2: Liquidity Crises



Notation: dotted line is $\theta_1(q, R)$, solid line is $1 - \beta(q, R)$.

Proposition 3. *If $\theta \leq \theta_1(q, R)$, then a unique equilibrium exists. This equilibrium is symmetric: each fund reserves a fraction of liquidity in amount of ℓ_a, where*

$$\ell_a = \frac{q(1-\theta)^2(1+R)^2 - qR^2 - R}{q(1-\theta)^2(1+R)^2 - qR^2}.$$

In the case of crisis the interest rate

$$r_a = 0.$$

The normalized expected final value of any fund equals v_a, where

$$v_a = \frac{q(2-q)(1-\theta)^2(1+R)^2 + (1-q)^2R^2}{2q(1-\theta)^2(1+R)^2 - 2qR^2}.$$

We summarize the analysis of the laissez-faire equilibrium in Fig. 2. The figure plots the three distinct parameter regions as functions of crisis probability q (for $R = 0.1$ and $R = 0.5$). When pledgeability among funds is relatively high, and the ex ante probability of crisis is low, we are in the severe region: funds specialize into either liquidity provision or long-term investment, and when a crisis occurs, short term interest rates peak. When interbank pledgeability is lower, or crises occur more often, we are in the symmetric normal region. If pledgeability is very low, this region turns into the autarky region, where the interbank market is no longer capable of efficiently redistributing available liquidity and funds are to some extent autarkic.

Fig. 3 plots the behaviour of aggregate equilibrium liquid reserves, short-term interest rates and fund expected values as either pledgeability θ or crisis probability q varies. We see that aggregate liquidity changes discontinuously when moving from the severe to the normal region. Indeed, on the boundary between those two regions, the ex ante expected value of an individual fund is independent of its liquidity, and funds are indifferent in their liquid holdings.

A second notable observation is that liquidity is non-monotonic in pledgeability θ. The explanation is that two opposing effects are at work. On the one hand, increasing pledgeability improves risk pooling through the interbank market, and this renders hoarding of liquid reserves less important. On the other hand, improved pledgeability increases demand for liquidity in the crisis state, increasing the return that those with excess liquidity reap from trading (as is manifest in the increasing short-term rate in the normal and severe regions).

5 Ex-ante Coordination: Minimum Liquidity Requirements

In this section we discuss the effects of introducing minimum liquidity requirements. We model these requirements by mandating funds to invest a minimum amount of $\bar{\ell}$ in liquid assets on date one. If there is a crisis on date two, then the funds are free to use these reserves at their discretion.[5]

We first show that in the normal region, which is characterized by the symmetric laissez-faire equilibrium, increasing ℓ relative to its laissez-faire level raises welfare. We have that in the symmetric equilibrium the total aggregate value of the surviving funds equals

$$S(\ell) = (1-q)\left((1+R)(1-\ell)+\ell\right) + q(1+R)(1-\ell)\bar{\varepsilon}(\ell).$$

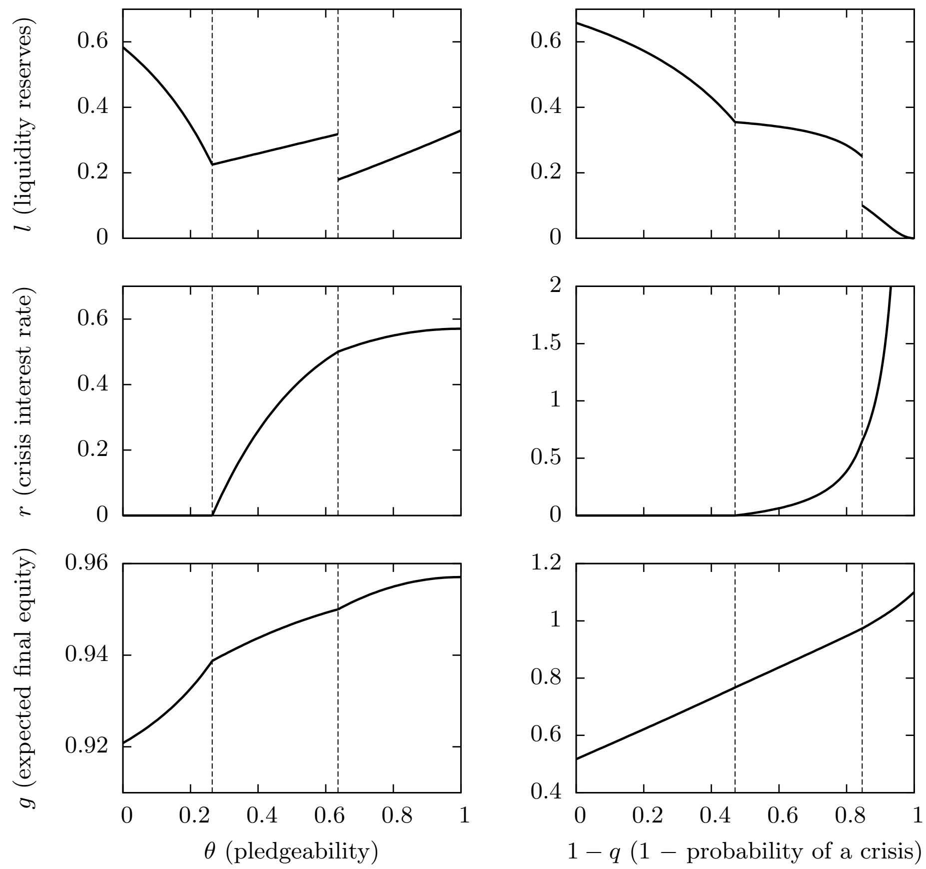
The first part in this expression denotes the value of funds in the absence of a shock. The second part equals the aggregate value of the surviving funds if there is a shock, where the surviving funds are those with $\varepsilon_i \leq \bar{\varepsilon}(\ell)$. Note that payments between funds are transfers, and there is no remaining liquidity left after the interbank market has cleared. If we use that the market for liquidity clears in the crisis state, i.e., $\bar{\varepsilon}(\ell) = 2\ell$, then optimizing over ℓ immediately gives

$$\ell^* = \frac{1}{2} - \frac{\beta}{4} > \ell_n,$$

where ℓ_n is the laissez-fair level of liquidity savings (see Proposition 1). Thus, the liquidity level that optimizes the value of surviving funds is larger than

[5]This modelling approach coincides with the view on minimum liquidity requirements as currently expressed by the Group of Governors and Heads of Supervision of the Basel Committee, see `http://www.bis.org/press/p120108.htm`.

Figure 3: Liquidity, Interest Rates and Value in Equilibrium



Initial parameters' values: $\theta = 0.5$, $q = 0.2$, $R = 0.1$. Notation: dotted lines denote regions' boundaries (see Fig. 2), with, from left to right, the autarky, normal and severe regions, respectively.

that realised in the laissez-faire equilibrium. Similarly, within the severe crisis region, which is characterized by the asymmetric equilibrium where funds save either maximum or zero liquidity, exogenously raising the fraction α of liquidity providers raises the value of surviving funds. Total expected value equals

$$S(\alpha) = (1-q)\left(\alpha + (1-\alpha)(1+R)\right) + q(1-\alpha)\bar{\varepsilon}(\alpha)(1+R),$$

where the second term reflects that in crisis all available remaining liquidity of the liquid funds is used to save the first $\bar{\varepsilon}$ of the illiquid funds. The fraction α of funds that invest in liquid assets supply liquidity in amount of $\alpha/2$, since on average one half is spent on their own liquidity shocks ε_i. In equilibrium this supply has to equal the demand for liquidity from illiquid funds, which amounts to $(1-\alpha)\bar{\varepsilon}^2/2$. We get

$$\bar{\varepsilon}(\alpha) = \sqrt{\frac{\alpha}{1-\alpha}}.$$

We find that at the optimum $\varepsilon^* = \bar{\varepsilon}(\alpha^*)$ satisfies

$$\varepsilon^* = -\beta + \sqrt{1+\beta^2} \geq \bar{\varepsilon}(r_s, \ell_i = 0) = \frac{-\beta + \sqrt{1+\beta^2-(1-\theta)^2}}{2-\theta},$$

where $\bar{\varepsilon}(r_s, \ell_i = 0)$ is the fraction of saved funds in the asymmetric laissez-faire equilibrium, see Equations (3) and (4), with equality only when when $\theta = 1$. Again, too little liquidity is saved from an ex ante welfare perspective.

This is summarized in the following proposition.

Proposition 4. *In the laissez-faire equilibrium with $\theta_1 < \theta < 1$, funds reserve suboptimal liquidity if we define welfare as the total value of surviving funds.*

The intuition for this result is that holding liquidity exhibits a positive externality, which funds do not take into account. Holding marginally more liquidity lowers the short-term interest rate in case of a crisis, thus saving some distressed but solvent funds that otherwise would have gone bankrupt due to partial pledgeability.

This observation suggests that there might be a scope for liquidity requirements to improve welfare. Indeed, we have already seen that in the normal region raising liquidity improves welfare. Requiring funds to hold marginally more liquidity achieves exactly that.

In the severe region, however, a fraction $1-\alpha$ of funds reserve zero liquidity. Consider a regulation that requires funds to hold minimum liquidity $\bar{\ell} > 0$ on date one. We will show that welfare in the severe region is decreasing in $\bar{\ell}$ at $\bar{\ell} = 0$.

A minimum $\bar{\ell}$ means that in the asymmetric equilibrium the $(1-\alpha)$ funds investing in illiquid projects can only invest up to $(1-\bar{\ell})$. The equilibrium fraction α will depend on $\bar{\ell}$, and will on the margin decrease in $\bar{\ell}$. Total welfare then becomes

$$S(\alpha(\bar{\ell}),\bar{\ell}) = (1-q)\left(\alpha(\bar{\ell}) + (1-\alpha(\bar{\ell}))\left[(1+R)(1-\bar{\ell})+\bar{\ell}\right]\right) + \\ q(1-\alpha(\bar{\ell}))\bar{\varepsilon}(\alpha(\bar{\ell}),\bar{\ell})(1+R)(1-\bar{\ell}).$$

We can compute the expression for $\bar{\varepsilon}(\alpha(\bar{\ell}),\bar{\ell})$ from market clearing in the crisis state,

$$\frac{1}{2}\alpha(\bar{\ell}) + (1-\alpha(\bar{\ell}))\int_0^{\bar{\ell}} d\varepsilon(\bar{\ell}-\varepsilon) = (1-\alpha(\bar{\ell}))\int_{\bar{\ell}}^{\bar{\varepsilon}} d\varepsilon(\varepsilon-\bar{\ell}),$$

where the left-hand side is the available supply of liquidity while the right hand side equals the demand from liquidity-poor funds. This yields

$$\bar{\varepsilon} = \bar{\ell} + \sqrt{\bar{\ell}^2 + \frac{\alpha(\bar{\ell})}{1-\alpha(\bar{\ell})}}.$$

Now we can compute $\frac{dS}{d\bar{\ell}} = \frac{\partial S}{\partial \alpha}\frac{\partial \alpha}{\partial \bar{\ell}} + \frac{\partial S}{\partial \bar{\ell}}$ and evaluate it at $\bar{\ell} = 0$. The first term is negative, as we already found that in competitive equilibrium, α is too low. For the second term, we have

$$\left.\frac{\partial S}{\partial \bar{\ell}}\right|_{\bar{\ell}=0} = q(1-\alpha)(1+R)\left[-\beta + 1 + \beta - \sqrt{1+\beta^2}\right] < 0,$$

so that at the margin, $\frac{dS}{d\bar{\ell}} < 0$.

The following proposition summarizes this discussion.

Proposition 5. *In the normal region, where $\theta_1 < \theta < 1 - \beta(q,R)$, setting minimum liquidity requirements marginally above ℓ_n improves welfare. In the severe region, where $1 - \beta(q,R) < \theta < 1$, obliging funds holding zero liquidity to hold marginally more liquidity reduces welfare.*

The intuition is that in the severe crisis region, liquidity regulation targets funds holding zero liquidity. Although forcing these funds to hold marginally more liquidity increases their probability of survival, it does so in an inefficient way because at the same time it reduces their value when they do survive. In contrast, raising the number of funds that save maximum liquidity would avoid this inefficiency.

6 Ex-post Intervention: Central Bank Policy

There are two assets in our model: cash and long-term investments. In the absence of central bank intervention the amount of cash in the economy stays constant. However, portfolio choices of funds influence how much cash resides on their books and how much cash circulates in the rest of the economy. Because long-term investments of funds are not pledgeable to outsiders, the only source of private liquidity on the interbank market on date two is the cash that is on the books of funds.

If there is a crisis on date two, then some solvent funds go bankrupt due to liquidity shortages. The central bank can reduce the number of these inefficient bankruptcies by using quantitative easing, i.e. by issuing new cash and lending it out on the interbank market.[6] Quantitative easing extends available credit, decreases the short-term interest rate and thus alleviates liquidity shortages.

How much of their investments distressed funds can pledge to the central bank in comparison with pledging them to other funds depends, generally speaking, on how the efficiency of monitoring compares between the central bank and the funds. Discussing comparative advantages and disadvantages of public monitoring versus private monitoring lies outside the scope of our paper. So, for simplicity, we assume that distressed funds face the same pledgeability constraint, namely constraint (2), no matter whether they borrow from other funds or from the central bank.

If the central bank commits to lower the interest rate through quantitative easing in case of a crisis, this commitment will induce a moral hazard problem. Anticipating help from the central bank in case of a crisis, the funds will decrease their cash reserves. Effectively, there will be crowding out of private liquidity with public liquidity. In this section we study this crowding out effect in a scenario, in which the central bank commits to marginally lower the interest rate in case of a crisis. A marginal perspective gives a first order approximation for the effects of central bank intervention, and it also simplifies the analysis.

We use the following notation: we put a hat on every variable in case there is contingent central bank intervention, and we leave the variables unhatted if we consider the laissez-faire equilibrium.

Suppose the central bank plans to *lower* the short-term interest rate in case of a crisis by $\Delta r > 0$ in comparison to its laissez-faire value. Let $\Delta L(\Delta r)$ denote the amount of cash that the central bank needs to issue on date two to achieve the set reduction in the interest rate. Then, if there is a risk of a normal crisis, total private liquidity decreases by $\ell_n - \hat{\ell}_n$ on date one. If the crisis indeed occurs, then the injection of public liquidity equals ΔL on date two. We define the conditional (given there has been a crisis)

[6] While we will focus of quantitative easing, primarily this is a restriction of our setup: we do not model government bonds. We do not expect our results to change qualitatively if we explicitly introduce governments bonds—for example as an asset that is fully pledgeable to insiders and not pledgeable to outsiders—and then look at open-market operations instead of quantitative easing.

crowding out effect κ_n as follows:

$$\kappa_n = \frac{\ell_n - \hat{\ell}_n}{\Delta L}.$$

If there is a risk of a severe crisis, then total private liquidity decreases by $\alpha - \hat{\alpha}$. In this case we define the conditional crowding out effect as

$$\kappa_s = \frac{\alpha - \hat{\alpha}}{\Delta L}.$$

We have

Proposition 6. *If $\theta_1(q, R) < \theta < 1 - \beta(q, R)$ (the normal region), then $\ell_n - \hat{\ell}_n \to 0$ and $\Delta L \to 0$ as $\Delta r \to 0$. Further,*

$$\kappa_n \to \frac{4 - 2(1-\theta)(1-\theta+\beta)}{(1-\beta+\theta)^2}.$$

If $\theta > 1 - \beta(q, R)$ (the severe region), then all private liquidity is crowded out, $\hat{\alpha} = 0$ as $\Delta r \to 0$ and

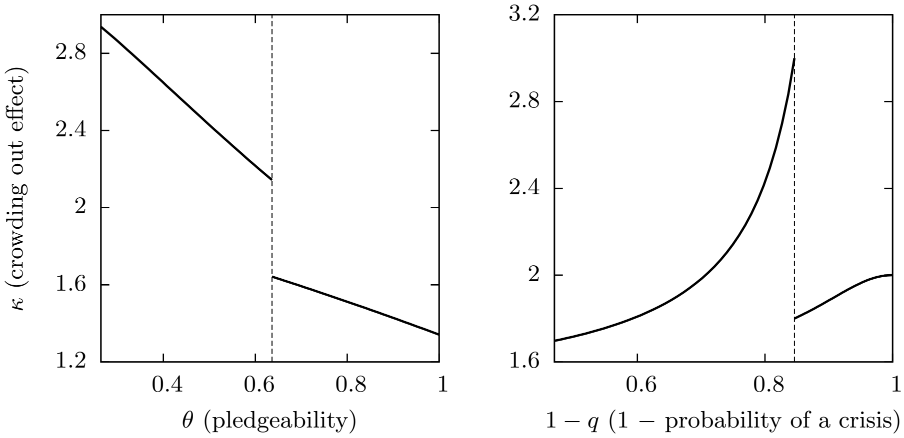
$$\kappa_s = \frac{\alpha}{\Delta L} \to 2(1-\alpha) = \frac{2(1+r_s)^2}{\theta^2(1+R)^2 + (1+r_s)^2}.$$

If there is a risk of a normal crisis, then—as we argue is Section (4)—the value function $v_n(\ell)$ is strictly concave in cash reserves ℓ. Consequently, each fund diversifies its portfolio to include both cash and investments. If the central bank commits to marginally lower the interest rate in case of a crisis, then hoarding cash becomes marginally less attractive. Due to the concavity of $v(\ell)$, the optimal cash reserves marginally decrease. However, if there is a risk of a severe crisis, then the value function $v_s(\ell)$ is strictly convex and has equal endpoints. Consequently, some funds are pure investors while others are pure liquidity providers. If due to a marginal intervention by the central bank hoarding cash becomes marginally less attractive than investing, i.e. if $v(0) < v(1)$, then due to the convexity of $v(\ell)$ all funds become investors and private cash reserves drop to zero.

Fig. 4 plots the crowding out effect for $\Delta r \to 0$. The value of cash reserves is comprised of two parts: insurance against liquidity shocks, and an option to profitably finance distressed funds. Central bank intervention is a substitute for the former role of cash reserves. As pledgeability increases, funds rely more on the interbank market to channel the support of the central bank, rather than on their cash reserves. Consequently, the central bank intervention causes a smaller crowding out of those reserves when pledgeability is higher.

If a crisis becomes less likely, then the opportunity costs of holding cash reserves increase. Therefore, funds are willing to substitute more of their private cash reserves for the public liquidity of the central bank.

Figure 4: Conditional Crowding Out Effect as $\Delta r \to 0$



Initial parameters' values: $\theta = 0.5$, $q = 0.2$, $R = 0.1$. Notation: dotted lines denote regions' boundaries (see Fig. 2).

Summarizing, crowding out effect decreases in pledgeability and in the probability of a crisis. So, if assets are relatively well-pledgeable, or if it is clear that a crisis might happen, then the commitment by the central bank to intervene in case of a crisis, and the subsequent crowding out of private liquidity with public liquidity due to moral hazard, is relatively less of an issue.

7 Conclusions

Liquidity shortage was a prominent aspect of the financial crisis of 2007–2009. A lot of policy responses ensued, ranging from quantitative easings aimed at restoring liquidity markets to revisiting banking regulations with the intention of guaranteeing a better pool of private liquidity in cases of future turmoil. While the simple line of argument that substantiates these policies is clear, a more comprehensive theoretical perspective is mostly lacking. Adhering to the notion of partial pledgeability that is advocated by Holmström and Tirole, we set up a basic model of liquidity markets that helps in such analysis. Our primarily goals were: (i) to study private incentives to reserve liquidity given that the interbank market can be relied upon for liquidity provision, (ii) to asses the moral hazard of the central bank's commitment to quantitative easings in case of crises, and (iii) to asses the effects of minimum liquidity requirements.

We showed that the distribution of private liquidity reserves can be of two types. Given relatively lower pledgeability and given a relatively higher probability of a crisis (the normal region, as we call it), all intermediaries reserve liquidity. However, if pledgeability is high or if the probability of a crisis is low (the severe region), then the distribution is bimodal. Some intermediaries reserve no liquidity and fully commit their assets to long-term investments, while others reserve to the maximum and, if there is a crisis, act as liquidity providers. This result is empirically testable, and as such can be used to validate our model.

We next showed that this segregation has policy implications. In the normal region a commitment by the central bank to marginally lower the interest rate in case of a crisis will cause a marginal decrease in private liquidity reserves, i.e. there will be a marginal crowding out of private liquidity with public liquidity. However, in the severe region the same marginal commitment will cause a complete crowding out of private liquidity. The central bank will be the only provider of liquidity if a crisis occurs. Further, in the normal region minimum liquidity requirements are welfare improving, but in the sever region minimum liquidity requirements can be detrimental to welfare. The latter occurs because symmetric minimum liquidity requirements distort the asymmetric distribution of private liquidity reserves. A regulation that raises the share of intermediaries that act as pure liquidity providers will be more effective in the severe region.

Generally speaking, we showed that the distribution of private liquidity reserves can be a deciding factor for the effectiveness of ex-ante and ex-post governmental interventions in the banking sector.

Appendix A

This appendix adopts the ideas of Stiglitz and Weiss (1981) to show how partial pledgeability can arise from agency costs of debt in our model.

Suppose that after date two the manager of a fund i has certain discretion over the risks and returns of the long-term investments of his fund. Namely, the manager can choose either a riskless continuation with a gross return $1 + R$ or a risky continuation, which yields $1 + \tilde{R} > 1 + R$ with probability $p < 1$ and zero with probability $1-p$. Suppose further that if there is no debt, then the manager prefers the riskless continuation, i.e. $1 + R > p(1 + \tilde{R})$.

In general, there are two markets: for risk-free debt, and for risky debt. The creditors (the funds with extra liquidity) are risk-neutral, therefore the expected returns have to be the same. So,

$$1 + r = p(1 + \tilde{r}),$$

where $\tilde{r}$ denotes the short-term interest rate on risky debt.

Let $I_i = (1 - \ell_i)E$ denote the long-term investments of a distressed fund i, and let $D_i = (\varepsilon_i - \ell_i)E$ denote the loan it needs to stay operational. The fund can obtain this loan at the risk-free rate r as long as its manager chooses the riskless continuation, i.e. as long as

$$(1+R)I_i - (1+r)D_i \geq p\big((1+\tilde{R})I_i - (1+r)D_i\big).$$

Rearranging the terms gives:

$$(1+r)D_i \leq \theta(1+R)I_i, \tag{5}$$

where

$$\theta = \frac{1}{1-p}\left(1 - p\frac{1+\tilde{R}}{1+R}\right).$$

Constraint (5) is precisely the credit constraint (2) that we assume in the main model.

If the fund i experiences a large enough shock ε_i, then (5) would not hold and the fund would not be able to refinance itself at the risk-free interest rate r. Suppose this is the case, i.e. suppose that

$$(1+r)D_i > \theta(1+R)I_i$$

or, using $1+r = p(1+\tilde{r})$,

$$(1+\tilde{r})D_i > \frac{\theta(1+R)}{p}I_i. \tag{6}$$

In this case the manager chooses the risky continuation. The manager can finance the risky continuation at the interest rate $\tilde{r}$ as long as

$$(1+\tilde{r})D_i \leq (1+\tilde{R})I_i. \tag{7}$$

For simplicity, in the main model we assume that the market for risky debt does not exist: a fund that can not satisfy (5) necessarily goes bankrupt. Comparing (6) and (7), we obtain the following condition that is necessary and sufficient to preclude the market for risky debt:

$$\frac{\theta(1+R)}{p} \geq (1+\tilde{R}).$$

Expanding and simplifying gives:

$$1+R \geq p(2-p)(1+\tilde{R}). \tag{8}$$

So, if we suppose that (8) holds, then this simple model with agency costs rationalizes the credit constraint (2). It is straightforward to further show that any $\theta \in \left(\frac{1}{2}, 1\right)$ is rationalizable (there exist corresponding p and $\tilde{R}$). In the main model we assume that $0 < \theta < 1$, because other factors such as higher discrepancy over continuation options, monitoring costs, private benefits, etc. can lower θ further down.

Appendix B

Proof of Propositions 1, 2 and 3. Let

$$Y = \{(q, \theta, R) \,|\, 0 < q < 1,\ 0 < \theta < 1,\ R > 0\}$$

denote the parameter space of our model. Further, let

$$\begin{aligned}
X_a &= Y \cap \{(q, \theta, R) \,|\, \theta \leq \theta_1(q, R)\}, \\
X_n &= Y \cap \{(q, \theta, R) \,|\, \theta_1(q, R) < \theta < \theta_2(q, R)\}, \\
X_s &= Y \cap \{(q, \theta, R) \,|\, \theta > \theta_2(q, R)\},
\end{aligned}$$

where

$$\begin{aligned}
\theta_1(q, R) &= 1 - \frac{\sqrt{(1+2q)^2 R^2 + 8qR} - R}{2q(1+R)}, \\
\theta_2(q, R) &= 1 - \beta(q, R) = 1 - \frac{(1-q)R}{q(1+R)}.
\end{aligned}$$

We have to show that for each X there is a unique equilibrium as given by the propositions. For completeness, let

$$X_0 = Y \cap \{(q, \theta, R) \,|\, \theta = \theta_2(q, R)\}.$$

We consider N ex-ante homogeneous funds, whose initial joint assets are normalized to 1. Define E as initial assets per fund, i.e. $NE = 1$. The following proof if for the case of $N \to \infty$.

With probability $1 - q$ there is no crisis, in which case $\varepsilon_i = 0$ for all i, there is zero demand for liquidity and $r = 0$. Consequently, the final equity of a fund i is $f(\ell_i)E$, where

$$f(\ell_i) = (1 + R)(1 - \ell_i) + \ell_i.$$

With probability q there is a crisis, in which case a fund i survives its liquidity shock on date two as long as

$$\theta(1 + R)(1 - \ell_i) \geq (1 + r_c)(\varepsilon_i - \ell_i), \tag{9}$$

where r_c is the crisis interest rate. Inequality (9) can be rewritten as $\varepsilon_i \leq \bar{\varepsilon}(\ell_i)$, where

$$\bar{\varepsilon}(\ell_i) = \ell_i + \frac{\theta(1 + R)(1 - \ell_i)}{1 + r_c}.$$

In the case of crisis the final equity of a solvent fund i is $f(\ell_i)E$, where

$$f(\ell_i) = (1 + R)(1 - \ell_i) + (1 + r_c)(\ell_i - \varepsilon_i),$$

The final equity of a bankrupt fund is zero.

In general, two cases are possible: (1) $\theta(1+R) < 1+r_c$ and (2) $\theta(1+R) \geq 1 + r_c$. We consider these cases in turn, starting from the first one.

Case (1) So, suppose $\theta(1+R) < 1+r_c$. Then $\bar{\varepsilon}(\ell_i) \leq 1$ for any $\ell_i \in [0,1]$. Having $\varepsilon_i \sim U[0,1]$, we obtain that the expected final equity of a fund i equals $v(\ell_i)E$, where

$$v(\ell_i) = \mathbb{E}f(\ell_i) = q\int_0^{\bar{\varepsilon}(\ell_i)} \big((1+R)(1-\ell_i) + (1+r_c)(\ell_i - \varepsilon_i)\big)\, d\varepsilon_i + \\ (1-q)\big((1+R)(1-\ell_i)+\ell_i\big) = \\ \frac{q\big((2-\theta)(1+R)(1-\ell_i) + (1+r_c)\ell_i\big)\big(\theta(1+R)(1-\ell_i) + (1+r_c)\ell_i\big)}{2(1+r_c)} + \\ (1-q)\big((1+R)(1-\ell_i)+\ell_i\big).$$

The manager of the fund i maximizes $v(\ell_i)$ in ℓ_i. For a fixed N the choice of ℓ_i influences the interest rate r_c, because an individual fund has a non-negligible influence on the interbank market. However, as N increases, this influence becomes smaller and in the limit equals zero.

Let

$$L = \arg\max_{\ell} v(\ell).$$

Because $v(\ell)$ is quadratic in ℓ, in general there can be five cases: (1.1) $v(\ell)$ is strictly concave and possesses an interior maximum, in which case set $L = \{\ell\}$ with $0 < \ell < 1$; (1.2) $v(\ell)$ is constant, in which case $L = [0,1]$; (1.3) $v(\ell)$ is strictly convex and has equal endpoints, in which case $L = \{0,1\}$; (1.4) $L = \{0\}$; and (1.5) $L = \{1\}$. Before we proceed with these cases, it is convenient to write down conditions for the concavity/convexity of $v(\ell_i)$. We have

$$\frac{d^2 v(\ell_i)}{dl_i^2} = q\left(\frac{\theta(1+R)}{1+r_c} - 1\right)\left(\frac{(2-\theta)(1+R)}{1+r_c} - 1\right).$$

So, if $\frac{1+r_c}{1+R} \in (-\infty, \theta) \cup (2-\theta, \infty)$, then $v(\ell_i)$ is strictly convex. If $\frac{1+r_c}{1+R} \in (\theta, 2-\theta)$, then $v(\ell_i)$ is strictly concave.

Case (1.1) Suppose $v(\ell_i)$ is strictly concave and has an interior maximum at $\ell_i = \ell$. Then each fund reserves ℓE units of liquidity. Suppose a crisis hits. If $\varepsilon_i \leq \ell$ for some fund i, then this fund supplies $\ell E - \varepsilon_i E$ units of liquidity to the interbank market as long as $r_c > 0$. Individual supply is perfectly inelastic, because there is no alternative use for liquidity on date two. Let $\mathbb{I}$ denote the indicator function. Using $NE = 1$ and using the fact that the shocks are independent, we obtain total supply of liquidity:

$$S = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{I}(\varepsilon_i \leq \ell)(\ell E - \varepsilon_i E) = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} \mathbb{I}(\varepsilon_i \leq \ell)(\ell - \varepsilon_i)}{N} NE = \\ \mathbb{E}\big(\mathbb{I}(\varepsilon_i \leq \ell)(\ell - \varepsilon_i)\big) = \int_0^{\ell} (\ell - \varepsilon)\, d\varepsilon = \frac{\ell^2}{2} \quad \text{for} \quad r_c > 0, \quad (10)$$

$S \in \left[0, \frac{\ell^2}{2}\right]$ for $r_c = 0$ and $S = 0$ for $r_c < 0$.

If $\ell < \varepsilon_i \leq \bar{\varepsilon}(\ell)$ for some fund i, then the fund borrows $\varepsilon_i E - \ell E$ units of liquidity. Summing up gives total demand for liquidity:

$$D = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{I}\big(\ell < \varepsilon_i \leq \bar{\varepsilon}(\ell)\big)(\varepsilon_i E - \ell E) = \int_{\ell}^{\bar{\varepsilon}(\ell)} (\varepsilon - \ell)\, d\varepsilon = \frac{1}{2}\left(\frac{\theta(1+R)(1-\ell)}{1+r_c}\right)^2. \quad (11)$$

Total demand is downward sloping, because with higher interest rates fewer funds can meet their credit constraint.

Trivially, $r_c < 0$ never clears the market. Suppose that $r_c = 0$ is the clearing interest rate. Solving the F.O.C. for the maximization of v and plugging in $r_c = 0$ gives

$$\ell = \frac{q(1-\theta)^2(1+R)^2 - qR^2 - R}{q(1-\theta)^2(1+R)^2 - qR^2} \quad (12)$$

and

$$v(\ell) = \frac{q(2-q)(1-\theta)^2(1+R)^2 + (1-q)^2R^2}{2q(1-\theta)^2(1+R)^2 - 2qR^2}.$$

This ℓ together with $r_c = 0$ gives an equilibrium as long as $0 < \ell < 1$ and $\frac{1+r_c}{1+R} = \frac{1}{1+R} \in (\theta, 2-\theta)$, because we are in Case (1.1); and as long as $S = D$ implies $r_c = 0$. The latter condition is satisfied if and only if

$$\frac{(\theta(1+R)(1-\ell))^2}{2} \leq \frac{\ell^2}{2}$$

or, equivalently,

$$\ell \geq \frac{\theta(1+R)}{\theta(1+R)+1}. \quad (13)$$

Let us now analyse these conditions. Given $R > 0$ and $\theta \leq 1$, the condition $\frac{1}{1+R} < 2-\theta$ is satisfied. Suppose $\frac{1}{1+R} \leq \theta$, then

$$(1-\theta)^2(1+R)^2 \leq R^2$$

and, consequently, $\ell > 1$. Thus, $\ell < 1$ implies $\frac{1}{1+R} > \theta$. Putting together (12) and (13) gives

$$q(1-\theta)^2(1+R)^2 - qR^2 - R \geq \theta R(1+R). \quad (14)$$

Trivially, (14) implies $0 < \ell < 1$. So, $\ell_i = \ell$ and $r_c = 0$ constitute an equilibrium as long as Condition (14) is satisfied, or equivalently, as long as

$$\theta \leq 1 - \frac{\sqrt{(1+2q)^2R^2 + 8qR} - R}{2q(1+R)}.$$

Additionally having that $0 < q < 1$, $0 < \theta < 1$ and $R > 0$, we get $(q, \theta, R) \in X_a$. At this moment in the proof we do not know if other equilibria are possible for $(q, \theta, R) \in X_a$, but later on we will see that all other equilibria require different (q, θ, R).

Current analysis is further restricted to the case when $\bar{\varepsilon}(\ell) \leq 1$. This restriction imposes no further inequalities on the parameters, because

$$\bar{\varepsilon}(\ell) = \ell + \theta(1 + R)(1 - \ell) < 1,$$

where the last inequality follows from $\frac{1}{1+R} > \theta$.

Suppose now that $r_c > 0$. Then $S = D$ implies

$$r_c = \frac{\theta(1 + R)(1 - \ell)}{\ell} - 1. \tag{15}$$

Writing down the F.O.C. for the maximization of v, plugging in (15) and solving for ℓ gives:

$$\ell = \frac{1 + \theta}{4} - \frac{(1 - q)R}{4q(1 + R)}. \tag{16}$$

Using (15), we further obtain that

$$v(\ell) = (1 - q)\big((1 + R)(1 - \ell) + \ell\big) + 2q(1 + R)(1 - \ell)\ell.$$

Liquidity reserves ℓ and the interest rate r_c, as defined by (15) and (16), constitute an equilibrium as long as $0 < \ell < 1$, $\frac{1+r_c}{1+R} \in (\theta, 2 - \theta)$, and $r_c > 0$. Having $q < 1$ and $\theta \leq 1$, we obtain from (16) that $\ell < \frac{1}{2}$. Consequently, $\frac{1+r_c}{1+R} > \theta$. So, the conditions $\ell < 1$ and $\frac{1+r_c}{1+R} > \theta$ are always satisfied. Given (15), the condition $\frac{1+r_c}{1+R} < 2 - \theta$ is equivalent to $\ell > \frac{\theta}{2}$, which implies $\ell > 0$. Expanding $\ell > \frac{\theta}{2}$ gives

$$\theta < 1 - \frac{(1 - q)R}{q(1 + R)}.$$

Finally, $r_c > 0$ if and only if

$$\ell < \frac{\theta(1 + R)}{\theta(1 + R) + 1}$$

or, equivalently,

$$\theta > 1 - \frac{\sqrt{(1 + 2q)^2 R^2 + 8qR} - R}{2q(1 + R)}.$$

Thus, ℓ and r_c as given by (16) and (15) constitute an equilibrium as long as $(q, \theta, R) \in X_n$.

Further,

$$\bar{\varepsilon}(\ell) = \ell + \frac{\theta(1 + R)(1 - \ell)}{1 + r_c} < 1,$$

because $\frac{1+r_c}{1+R} > \theta$.

Case (1.2) Suppose $v(\ell_i)$ is constant. Then the coefficients in front of ℓ_i^2 and ℓ_i have to equal zero. Solving the respective equations under the restriction that $q < 1$ gives:

$$\theta = 1 - \frac{(1-q)R}{q(1+R)}.$$

Together with $0 < q < 1$, $0 < \theta < 1$ and $R > 0$ we have $(q, \theta, R) \in X_0$.

Propositions 3, 1 and 2 do not cover $(q, \theta, R) \in X_0$, therefore further analysis of this case is unnecessary.[7]

Case (1.3) Suppose $v(\ell_i)$ is strictly convex and has equal endpoints, i.e. $v(0) = v(1)$. Then a fund i chooses either $\ell_i = 1$ or $\ell_i = 0$. Let α denote the share of funds that choose $\ell_i = 1$, i.e.

$$\alpha = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} \mathbb{I}(\ell_i = 1)}{N}.$$

The funds with $\ell_i = 1$ form the supply of liquidity, while those with $\ell_i = 0$ form the demand. Similarly to (10) and (11) we obtain

$$\begin{aligned} S = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{I}(\ell_i = 1)(E - \varepsilon_i E) = \\ \lim_{N\to\infty} \frac{\sum_{i=1}^{N} \mathbb{I}(\ell_i = 1)}{N} \cdot \frac{\sum_{i=1}^{N} \mathbb{I}(\ell_i = 1)(1-\varepsilon_i)}{\sum_{i=1}^{N} \mathbb{I}(\ell_i = 1)} NE = \\ \alpha \mathbb{E}(1-\varepsilon_i) = \alpha \int_0^1 (1-\varepsilon)\, d\varepsilon = \frac{\alpha}{2} \quad \text{for} \quad r_c > 0, \end{aligned}$$

$S \in \left[0, \frac{\alpha}{2}\right]$ for $r_c = 0$, $S = 0$ for $r_c < 0$, and

$$\begin{aligned} D = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{I}(\ell_i = 0)\mathbb{I}(\varepsilon_i \le \bar{\varepsilon}(0)) \cdot \varepsilon_i E = (1-\alpha) \int_0^{\bar{\varepsilon}(0)} \varepsilon\, d\varepsilon = \\ \frac{1-\alpha}{2} \left(\frac{\theta(1+R)}{1+r_c} \right)^2. \end{aligned} \tag{17}$$

Clearly, $r_c < 0$ does not clear the market. Having $r_c \ge 0$ and solving for $v(0) = v(1)$ gives

$$r_c = \frac{(1-q)R}{q} + \sqrt{\left(\frac{(1-q)R}{q} \right)^2 + \theta(2-\theta)(1+R)^2} - 1 \tag{18}$$

[7] In this case multiple equilibria are possible, because each fund i is indifferent between any $\ell_i \in [0, 1]$. However, this case has measure zero in our parameter space, therefore we have chosen to omit it from the propositions.

and

$$v(0) = v(1) = \frac{q(1+r_c)}{2} + (1-q).$$

This Case (1.3) constitutes an equilibrium as long as $r_c \geq 0$ and $\frac{1+r_c}{1+R} \in (-\infty, \theta) \cup (2-\theta, \infty)$. Suppose $\frac{1+r_c}{1+R} < \theta$. Solving this inequality gives that

$$\theta > 1 + \frac{(1-q)R}{q(1+R)} > 1.$$

As $\theta \leq 1$, the only remaining possibility is that $\frac{1+r_c}{1+R} > 2-\theta$. Solving this second inequality we obtain

$$\theta > 1 - \frac{(1-q)R}{q(1+R)}.$$

Trivially, $\frac{1+r_c}{1+R} > 2-\theta$ further implies that $r_c > 0$. Then, from $S = D$, we get

$$\alpha = \frac{\theta^2(1+R)^2}{\theta^2(1+R)^2 + (1+r_s)^2}. \tag{19}$$

Clearly, $0 < \alpha < 1$.

Thus, there is an equilibrium with a share α of funds having $\ell_i = 1$, the rest having $\ell_i = 0$, and with the interest rate r_c as given by (18) as long as $(q, \theta, R) \in X_s$.

Lastly, $\bar{\varepsilon}(1) = 1$ and

$$\bar{\varepsilon}(0) = \frac{\theta(1+R)}{1+r_c} < \frac{\theta}{2-\theta} \leq 1.$$

Case (1.4) Suppose $v(\ell_i)$ attains a unique maximum at $\ell_i = 0$. Then the supply of liquidity is zero, and the demand for liquidity is

$$D = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{I}(\varepsilon_i \leq \bar{\varepsilon}(0)) \cdot \varepsilon_i E = \frac{1}{2}\left(\frac{\theta(1+R)}{1+r_c}\right)^2.$$

There is no clearing r_c and, consequently, there is no equilibrium in this case.[8]

Case (1.5) Suppose $v(\ell_i)$ attains a unique maximum at $\ell_i = 1$, i.e. $v(1) > v(\ell_i)$ for all $\ell_i < 1$. Then the demand for liquidity is zero and $r_c = 0$.

[8]Less formally, if a single fund chooses $\ell_i = 1$, then it almost always ends up with excess liquidity and can charge $r_c \to \infty$ on its loans. Consequently, $\ell_i = 0$ can never deliver a maximum to $v(\ell_i)$.

Expanding and simplifying $v(1) - v(\ell_i) > 0$, and using $r_c = 0$, we obtain that

$$q(\theta(1+R)-1)((2-\theta)(1+R)-1)\cdot \ell_i + \\ q\theta(\theta-2)(1+R)^2 + 2qR - 2R + q > 0 \quad (20)$$

for all $\ell_i < 1$.

Suppose $\theta(1+R) \geq 1$. Then Inequality (20) holds for all $\ell_i < 1$ if and only if it holds for $\ell_i = 0$. We have

$$q\theta(\theta-2)(1+R)^2 + 2qR - 2R + q > 0$$

or, using $\theta \leq 1$,

$$\theta < 1 - \frac{\sqrt{q^2R^2 + 2qR}}{q(1+R)}. \quad (21)$$

Inequality (21) together with $\theta(1+R) \geq 1$ implies $2qR < 0$, which is never satisfied.

Suppose $\theta(1+R) < 1$. Then Inequality (20) holds for all $\ell_i < 1$ if and only if it holds non-strictly for $\ell_i \to 1$. We get $-2R \geq 0$, which is never satisfied.

Summarizing, Case (1.5) does not deliver an equilibrium.

Case (2) The preceding analysis has been for the case when $\theta(1+R) < 1 + r_c$. Suppose instead that $\theta(1+R) \geq 1 + r_c$. It follows that $\bar{\varepsilon}(\ell_i) \geq 1$ for any $\ell_i \in [0,1]$, and we have

$$v(\ell_i) = \mathbb{E}f(\ell_i) = q\int_0^1 \big((1+R)(1-\ell_i) + (1+r_c)(\ell_i - \varepsilon_i)\big)\,d\varepsilon_i + \\ (1-q)\big((1+R)(1-\ell_i) + \ell_i\big) = \\ (R - qr_c)(1-\ell_i) + 1 - \frac{q(1-r_c)}{2}.$$

As $\theta(1+R) \geq 1 + r_c$, $\theta < 1$ and $q < 1$, we obtain that $R > qr_c$. Therefore $\ell_i = 0$ maximizes $v(\ell_i)$ for any i, and we are back in Case (1.4), in which there was no equilibrium.

So far we have analysed all possible cases. Each equilibrium that we have found exists if and only if (q, θ, R) belongs to a particular X. Thus, if sets X are disjoint, then each X delivers a unique equilibrium.

Trivially, $X_a \cap X_n = \emptyset$, $X_n \cap X_s = \emptyset$, $X_n \cap X_0 = \emptyset$, and $X_s \cap X_0 = \emptyset$. Consider $X_a \cap X_s$. If this set is non-empty, then

$$\begin{cases} \theta_1 > \theta_2, \\ \theta_1 > 0. \end{cases}$$

Expanding and rearranging terms gives:

$$\begin{cases} R - 2qR - q > 0, \\ R - 2qR - q < 0. \end{cases}$$

Thus, $X_a \cap X_s = \emptyset$. In a similar fashion we obtain that $X_a \cap X_0 = \emptyset$. $\square$

Proof of Proposition 6. If there is no central bank intervention, then in case of a crisis the short-term interest rate equals $r_c(q, \theta, R)$. The proposition is concerned with the following central bank intervention. Given (q, θ, R), the central banks commits on date one to inject enough liquidity into the interbank market on date two in case there is a crisis so as to achieve an interest rate of $r_c(q, \theta, R) - \Delta r$, where $\Delta r > 0$. Further, we will look at the consequences of this commitment in the limit, i.e. as $\Delta r \to 0$.

This proof borrows extensively from the proof of Propositions 1, 2 and 3, and we omit explicit references to that earlier proof so as not to clutter the exposition.

We know that $\theta(1+R) < 1+r_c$. Therefore there exists $\phi_1 > 0$ such that $\theta(1+R) < 1+r_c - \Delta r$ for any $\Delta r \leq \phi_1$. So, consider $\Delta r \leq \phi_1$. In this case $\bar{\varepsilon}(\ell_i) \leq 1$ for any $\ell_i \in [0, 1]$ and hence the expected final equity of a fund i is given by $v(\ell_i)E$, where

$$\begin{aligned} v(\ell_i) = \frac{1}{2(1+r_c-\Delta r)} \cdot q\big((2-\theta)(1+R)(1-\ell_i) + (1+r_c-\Delta r)\ell_i\big) \cdot \\ \big(\theta(1+R)(1-\ell_i) + (1+r_c-\Delta r)\ell_i\big) + (1-q)\big((1+R)(1-\ell_i) + \ell_i\big) \end{aligned} \quad (22)$$

Let us first consider the case when $(q, \theta, R) \in X_n$. If $\Delta r = 0$, then in this case F.O.C. deliver the unique and interior maximum of $v(\ell_i)$, namely ℓ_n. Because $v(\ell_i)$ is differentiable in ℓ_i and Δr, we have that there exists $\phi_2 > 0$ such that F.O.C. still deliver the unique and interior maximum of $v(\ell_i)$ if $\Delta r \leq \phi_2$. So, consider $\Delta r \leq \min\{\phi_1, \phi_2\}$, and denote the aforementioned maximum with $\hat{\ell}_n(\Delta r)$. Writing down the first order approximation gives:

$$\hat{\ell}_n(\Delta r) = \hat{\ell}_n(0) + \frac{d\hat{\ell}_n}{d\Delta r}(0) \cdot \Delta r + o(\Delta r),$$

or, using $\hat{\ell}_n(0) = \ell_n$,

$$\ell_n - \hat{\ell}_n(\Delta r) = -\frac{d\hat{\ell}_n}{d\Delta r}(0) \cdot \Delta r + o(\Delta r).$$

Given (22) and having $r_c = r_n$, because $(q, \theta, R) \in X_n$, we obtain after some algebra that

$$\frac{d\hat{\ell}_n}{d\Delta r}(0) = \frac{(1-q-\beta q)(1-\beta+\theta)^2\big((1-\theta)(1-\theta+\beta)-2\big)}{8\theta(1-q)(1-\theta+\beta)(1-\theta-\beta)},$$

where, as before,

$$\beta = \frac{(1-q)R}{q(1+R)}.$$

To achieve the promised interest rate $r_n - \Delta r$, the central bank needs to inject just enough cash to cover the excess demand for cash at that interest rate. Given private cash reserves of $\hat{\ell}_n E$ and following (10) and (11) we obtain:

$$\Delta L(\Delta r) = D - S = \frac{1}{2}\left(\frac{\theta(1+R)\big(1-\hat{\ell}_n(\Delta r)\big)}{1+r_n-\Delta r}\right)^2 - \frac{\hat{\ell}_n(\Delta r)^2}{2}.$$

If $\Delta r = 0$, then the market clears by definition, i.e. $L(0) = 0$. Hence

$$\Delta L(\Delta r) = \frac{d\Delta L}{d\Delta r}(0)\cdot \Delta r + o(\Delta r).$$

After some algebra we arrive at

$$\frac{d\Delta L}{d\Delta r}(0) = \frac{(1-q-\beta q)(1-\beta+\theta)^4}{16\theta(1-q)(1-\theta+\beta)(1-\theta-\beta)}.$$

So,

$$\kappa_n = \frac{\ell_n - \hat{\ell}_n(\Delta r)}{\Delta L(\Delta r)} \to -\frac{d\hat{\ell}_n}{d\Delta r}(0)\Big/\frac{d\Delta L}{d\Delta r}(0) = \frac{4-2(1-\theta)(1-\theta+\beta)}{(1-\beta+\theta)^2}$$

as $\Delta r \to 0$.

Let us now consider the case when $(q,\theta,R) \in X_s$. We have $r_c = r_s$. Further, we know that if $\Delta r = 0$, then in this case $\frac{1+r_s}{1+R} > 2-\theta$ and, consequently, $v(\ell_i)$ is strictly convex. Therefore there exists $\phi_3 > 0$ such that $\frac{1+r_s-\Delta r}{1+R} > 2-\theta$ and $v(\ell_i)$ is strictly convex for all $\Delta r \le \phi_3$. So, we will consider $\Delta r \le \min\{\phi_1,\phi_3\}$.

As $v(\ell_i)$ is strictly convex, it attains its maximum at one of the endpoints. Let $f(\Delta r) = v(0;\Delta r) - v(1;\Delta r)$. In the laissez-faire equilibrium the endpoints are equal, i.e. $f(0) = 0$. We have

$$\frac{df}{d\Delta r}(0) = \frac{q}{2}\left(\frac{\theta(2-\theta)(1+R)^2}{(1+r_s)^2} + 1\right) > 0.$$

So, there exists $\phi_4 > 0$ such that $v(0;\Delta r) > v(1;\Delta r)$ for $\Delta r \le \phi_4$. We further restrict our focus to $\Delta r \le \min\{\phi_1,\phi_3,\phi_4\}$.

As before, $\Delta L = D - S$. Given that $v(0) > v(1)$, all funds choose $\ell_i = 0$, i.e. $\hat{\alpha} = 0$. Then, in case of a crisis, the supply of cash is zero and, following (17), we get

$$\Delta L(\Delta r) = D = \frac{1}{2}\left(\frac{\theta(1+R)}{1+r_s-\Delta r}\right)^2 \to \frac{1}{2}\left(\frac{\theta(1+R)}{1+r_s}\right)^2$$

as $\Delta r \to 0$.

Given (19), we obtain that

$$\kappa_s = \frac{\alpha - \hat{\alpha}(\Delta r)}{\Delta L(\Delta r)} = \frac{\alpha}{\Delta L(\Delta r)} \to \frac{2(1+r_s)^2}{\theta^2(1+R)^2 + (1+r_s)^2} = 2(1-\alpha)$$

as $\Delta r \to 0$. □

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