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Abstract in Dutch

Dit onderzoek analyseert de relatie tussen flexibele uittreding - de mogelijkheid om individueel de datum te kiezen waarop men stopt - en spaar- en beleggingsgedrag over de levensloop. Het laat zien dat flexibele uittreding niet altijd resulteert in een grotere vraag naar risicovolle beleggingen, zoals vaak wordt aangenomen.

In de economische literatuur wordt vaak gesteld dat flexibiliteit in de uittreedbeslissing een belangrijk instrument is om financiële schokken op te vangen. Deze 'uitlaatklep' aan het einde van de loopbaan stelt mensen in staat om gedurende het actieve leven meer beleggingsrisico te nemen waardoor extra geprofiteerd kan worden van de risicopremie op risicodragende activa. Dit artikel laat echter zien dat de positieve samenhang tussen flexibele uittreding en risicovol beleggen minder - of zelfs negatief - wordt als niet alleen financiële risico's maar ook productiviteitsrisico's worden meegenomen in de analyse. Een productiviteitsschok beinvloedt in principe zowel de beloning van kapitaal als die van arbeid. Als de productiviteit onverwacht daalt, leidt dit dus niet alleen tot lagere aandelenrendementen maar ook tot lagere lonen. Dat betekent dat de prikkel om langer door te werken kleiner wordt. Flexibele uittreding biedt dus een minder goede bescherming tegen schokken met als gevolg dat de bereidheid om risicovol te beleggen daalt.

De studie toont verder aan dat de samenhang tussen flexibele uittreding en risicovol beleggen sterk afhangt van de vraag of individuen consumptie van goederen en vrije tijd als substituten, dan wel als complementaire goederen beschouwen. In het eerste geval zal men sneller besluiten de arbeidsmarkt te verlaten als lonen (de prijs van vrije tijd) laag zijn ten gevolge van een negatieve productiviteitsschok. Deze grotere gevoeligheid van de uittreedbeslissing voor veranderingen in de loonvoet vermindert de bescherming die flexibele uittreding biedt tegen onverwachte schokken en verlaagt daardoor de bereidheid risicovol te beleggen. Tot slot laat de studie zien dat algemene-evenwichtseffecten de verzekering van flexibele uittreding tegen schokken zowel kunnen vergroten als verkleinen, afhankelijk van de mate van substitutie tussen consumptie en vrije tijd.

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Retirement Flexibility and Portfolio Choice*

Yvonne Adema† Jan Bonenkamp‡ Lex Meijdam§

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Abstract

This paper explores the interaction between retirement flexibility and portfolio choice in an overlapping-generations model. We analyse this interaction both in a partial-equilibrium and general-equilibrium setting. Retirement flexibility is often seen as a hedge against capital-market risks which justifies more risky asset portfolios. We show, however, that this positive relationship between risk taking and retirement flexibility is weakened − and under some conditions even turned around − if not only capital-market risks but also productivity risks are considered. Productivity risk in combination with a high elasticity of substitution between consumption and leisure creates a positive correlation between asset returns and labour income, reducing the willingness of consumers to bear risk. Moreover, it turns out that general-equilibrium effects can either increase or decrease the equity exposure, depending on the degree of substitutability between consumption and leisure.

Key words: retirement (in)flexibility, portfolio allocation, risk, intratemporal substitution elasticity

JEL codes: E21, G11, J26

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1 Introduction

In many western countries, pension schemes typically move from contracts with high implicit tax rates and therefore predominantly inflexible retirement ages towards more actuarially neutral arrangements with flexible retirement ages.[1] This move to flexible pension schemes is partly forced by population ageing and the financial crisis which put the traditional social security schemes under financial pressure. Another important factor is the ongoing process of individualization and the resulting acknowledgement that individuals differ in their tastes for leisure, earnings capacities, wealth positions, and therefore have different preferences for retirement.

In this paper, we raise the question how this trend from inflexible to flexible pension contracts will affect consumption and portfolio decisions during working life. As stressed in the literature, the important advantage of retirement flexibility is that it provides insurance against all types of risks, like disability risk (Diamond and Mirrlees, 1978, 1986) or stock-market risk (Pestieau and Possen, 2009). The general idea of flexible retirement is that it gives individuals the ability to adjust working life to their own preferences and to avoid abrupt changes in life-time consumption. Viewed in this way, retirement flexibility serves as a hedge against adverse investment outcomes which allows for more risk taking in pension assets (see e.g. Bodie et al., 1992). The basic mechanism behind this result is the negative correlation between asset returns and labour income due to wealth effects in the retirement decision. Indeed, a negative wealth shock causes the marginal utility from leisure to decrease and hence agents increase labour supply which, in turn, raises labour income. Our analysis reveals that factors like the type of risk, the willingness of consumers to substitute consumption for leisure, and general-equilibrium effects have an important impact on the insurance provided by retirement flexibility. Different positions about these factors may change existing views from the literature.

The number of studies that focus on the interaction between portfolio, consumption and retirement decisions is rather limited. Starting point is the seminal paper of Bodie et al. (1992) which analyses this interaction assuming that labour can be adjusted continuously. Subsequent studies, like Choi and Shim (2006), Choi et al. (2008), Farhi and Panageas (2007) and Lachance (2003, 2004), model optimal retirement as a discretionary stopping problem. Although all these studies differ in many respects, they have in common that they use partial-equilibrium models and mainly stick to capital-market risks. In addition, they all find that more flexibility in the retirement decision increases the portfolio share invested in stocks.

[1] See van Vuuren (2011) for an extensive overview of recent trends in flexible retirement.

Compared to the existing literature in general and the work of Bodie et al. (1992) in particular, we add three important elements to the analysis on portfolio choice and retirement. First, we complement the partial-equilibrium approach with a general-equilibrium one. A general-equilibrium perspective seems the most natural road to take because the move to flexible pensions clearly is an international phenomenon. With general equilibrium, we explicitly recognize that consumption and labour supply decisions affect factor prices which, in turn, influence the insurance effect of retirement flexibility. To illustrate, if every old worker decides to work longer after an adverse shock, wages will decline making the insurance of retirement flexibility less effective. Second, we distinguish between productivity and depreciation risk and these risk factors are directly linked to production. This distinction is important because both risk factors constitute a rather different effect on income and substitution effects in labour supply. As will be shown, the relative strength of income and substitution effects determines whether retirement flexibility indeed serves as a hedge against poor asset returns. Third, following Choi et al. (2008), we allow for more general preferences which are characterized by a constant elasticity of substitution (CES) function of consumption and leisure. This specification allows the elasticity of substitution between labour and leisure to take any positive number.

To analyse the interaction between portfolio choice, consumption and retirement decisions, we develop a two-period overlapping-generations (OLG) model of a closed economy in the spirit of Samuelson (1958) and Diamond (1965). The model includes government debt and incorporates endogenous retirement. In our framework, the young working generation decides upon his consumption and the allocation of his asset portfolio. Agents can either invest in risk-free government bonds or in risky firm stocks. Our model is related to the model of Adema (2008) which is also a stochastic two-period OLG model of a closed economy with government debt. There, however, the return on bonds is subject to inflation risk while retirement is exogenous. In our model, retirement is endogenous and we compare two different retirement settings: under *flexible* retirement, the old generation can freely postpone or advance retirement in the second period after a realization of shocks; under *fixed* retirement, this generation has to make this decision already before shocks are revealed. Once set, this decision cannot be subsequently changed when new information becomes available.

We use log-linearization techniques to characterize the main insights of the model. This method is widely applied in the real business cycle literature (see e.g. Campbell, 1994; King et al., 2002 or Uhlig, 1999), but it is also often used in stochastic overlapping generations models (see Bohn, 2009; Jensen and Jørgensen, 2008 or Matsen and

Thøgersen, 2004). The standard procedure used in these studies is to first derive the non-stochastic steady state and then to take first-order Taylor approximations around this steady state. The resulting system of log-linear difference equations can then be solved either numerically or analytically. To study macroeconomic dynamics, as most of the aforementioned studies do, this procedure is sufficient. It is less suitable, however, for an analysis involving asset-pricing issues, as we do here. We therefore log-linearize the model around a stochastic steady state which explicitly takes the second-order risk terms into account. This method has already been used by Beetsma and Bovenberg (2009) and Bovenberg and Uhlig (2008), who both study risk-sharing issues in relation to social security, but until now it has never been applied to portfolio allocation in relation to endogenous retirement.

Our analysis provides some interesting insights. First, the positive relation between retirement flexibility and a higher risk appetite is weakened − and under some conditions even turned around − if not only depreciation shocks but also productivity shocks are considered. Depreciation shocks mainly affect the return on capital and through the income effect these shocks contribute to the traditional view that retirement flexibility increase risk-taking behaviour. Productivity shocks, in contrast, do not only affect capital returns but also influence wages. Consequently, productivity shocks also induce substitution effects in labour supply which work in the opposite direction. These substitution effects generate a positive correlation between asset returns and labour income, thereby reducing the risk-bearing capacity of consumers.

Second, confining the analysis to Cobb-Douglas utility, as most of the existing studies do, ignores the essential role of the elasticity of substitution between consumption and leisure in studying retirement flexibility. This elasticity of substitution governs the relative strength of income and substitution effects in labour supply and, hence, determines the insurance provided by retirement flexibility. Our analysis clearly shows that flexible retirement amplifies consumption volatility if substitution effects are important, a notion also put forward by Basak (1999).

Finally, we find that general-equilibrium effects play an important role in the interaction between portfolio choice and retirement. Ignoring these effects by sticking to a partial-equilibrium framework can either overstate or understate the hedging effect of retirement flexibility, dependent on the willingness of consumers to substitute between consumption and leisure. If the elasticity of substitution is high, agents choose to supply less labour after a negative productivity shock. In general equilibrium, this labour supply response exacerbates the direct fall in the return on capital due to the productivity

contraction. Compared to partial equilibrium, this higher sensitivity of the capital return for productivity risk results in lower portfolio shares invested in equity. Of course, for low elasticities of substitution just the opposite holds: then the insurance effect is more effective in general than in partial equilibrium, leading to higher equity shares.

The results of this paper are relevant for private or public pension institutions, like corporate pension funds, trust funds or life-insurance companies, to which individuals have dedicated or will dedicate their saving and investment decisions. As the development towards tailor-made pension products is still an ongoing process in many countries, the acknowledgement that investment policy should be based on individual preferences for retirement will become increasingly important. Even if individuals are able to make the retirement decision conditional on future states, our analysis shows that risky investment strategies are not always in their interest. This is in particular the case if shocks to pension wealth and wages are positively correlated or if consumers view leisure and consumption as close substitutes. Of course, whether substitution effects are important or not is largely an empirical question. In this respect, empirical studies have shown that implicit taxes have a large negative effect on the labour supply of elderly indicating that substitution effects are indeed important in retirement behaviour (Asch et al., 2005; Coile and Gruber, 2001 and Gruber and Wise, 2004). Moreover, many empirical studies exploring the impact of a change in pension wealth on the retirement decision find modest effects (Bloemen, 2010; French, 2005, and Krueger and Pischke, 1992).

The rest of this paper is organized as follows. Section 2 sets out the basics of the stochastic OLG model. In Section 3, we explain how to solve this model using a log-linearization technique around the stochastic steady state. Section 4 presents analytical results for a simplified model version of the model that reproduces the main findings of the current literature. In Section 5 we present and compare numerical results for the partial-equilibrium model and for the general-equilibrium model. Finally, Section 6 concludes the paper.

2 The model

In this section, we develop a two-period OLG model of a closed economy. In order to analyse the interaction between retirement and portfolio choice, we include government debt in the model as an alternative investment vehicle for future consumption and introduce endogenous retirement in the second period of life. The economy is subject to productivity risk and depreciation risk.

At each point in time, the young individual determines consumption of a single good and the proportion of financial wealth to invest in firm stocks. The old generation decides which fraction of the second period it will spend on working and on enjoying retirement. Following Bodie et al. (1992), we consider two different retirement settings: (i) under *flexible* retirement, the old generation can freely postpone or advance retirement in the second period after a realization of shocks; (ii) under *fixed* retirement, the retirement decision has to be made before shocks are revealed. Once set, the retirement age cannot be subsequently changed after new information has become available. Whatever the retirement setting (flexible or fixed), an individual sets his decision variables optimally, conditional on his information to date: his current financial wealth, the future dynamics of the asset returns and his uncertain future wage.

2.1 Production

The young and old generation are composed of the same large number of individuals and this number is normalized to unity. Production per young worker is described by a standard neoclassical constant-returns-to-scale Cobb-Douglas production function:

$$f(k_t, z_t) = A_t k_t^{\alpha} (1 + z_t)^{1-\alpha} \tag{1}$$

with A_t the stochastic total productivity parameter, α the capital share in production and k_t the capital stock per young worker. Total labour supply, $1 + z_t$, consists of young workers inelastically supplying one unit of labour and old workers, each spending a fraction $0 \leq z_t \leq 1$ of time on working. Profit maximisation and perfect competition among producers results in the standard equilibrium conditions:

$$w_t = (1 - \alpha) A_t k_t^{\alpha} (1 + z_t)^{-\alpha} \tag{2}$$

$$r_{k,t} + \delta_t = \alpha A_t k_t^{\alpha - 1} (1 + z_t)^{1-\alpha} \tag{3}$$

where w_t is the real wage, $r_{k,t}$ the return on capital and δ_t can be interpreted as the stochastic depreciation rate of capital.

Production and capital investment are important in this context because they endogenize the correlation between capital and labour income. Note that productivity risk directly affects the capital return and the wage rate, while depreciation risk only directly affects the return on capital. Of course, there is an indirect link between the wage rate and depreciation risk, to the extent that labour supply behaviour affects factor prices in

general equilibrium. Stochastic depreciation not only breaks down the (perfect) correlation between wages and capital returns, it also increases return volatility and may give capital returns a higher one-period-ahead variance than wages. The stochastic processes for total factor productivity and capital depreciation are:

$$\log A_t = \log A + \omega_{A,t} \tag{4}$$

$$\log \delta_t = \log \delta + \omega_{\delta,t} \tag{5}$$

with $\omega_{A,t}$ and $\omega_{\delta,t}$ independently and identically distributed with mean zero and variance σ_A^2 and σ_δ^2.

2.2 Consumers

Individuals derive utility from consumption and leisure. Expected life-time utility of a representative individual born at t is given by the following constant-relative-risk-aversion (CRRA) utility function:

$$U_t = \frac{c_{1,t}^{1-\theta} - 1}{1-\theta} + \beta \frac{\mathrm{E}_t\, v(c_{2,t+1}, 1 - z_{t+1})^{1-\theta} - 1}{1-\theta} \tag{6}$$

where $c_{1,t}$ is consumption when young at time t, $c_{2,t+1}$ is consumption when old at $t+1$, β is the time discount factor and θ is the coefficient of relative risk aversion which is identical to the inverse of the intertemporal elasticity of substitution. The per-period utility function $v(\cdot)$ has a CES specification and is defined as:[2]

$$v(c_2, 1-z) = \left[(1-\gamma)c_2^{1-\rho} + \gamma\,(1-z)^{1-\rho}\right]^{\frac{1}{(1-\rho)(1-\gamma)}} \tag{7}$$

where γ defines the relative preference for leisure and ρ represents the inverse of the elasticity of substitution between consumption and leisure in the second period. This specification includes the familiar Cobb-Douglas period utility function $v(c_2, 1-z) = c_2(1-z)^{\gamma/(1-\gamma)}$ if $\rho = 1$.

People can either invest in firm stocks which yield the stochastic return $r_{k,t+1}$ or in government bonds with the risk-free return $r_{b,t+1}$. The share of savings that is invested in stocks is denoted by λ_t, so that the return on the asset portfolio can be defined as:

$$r_{t+1} \equiv (1-\lambda_t) r_{b,t+1} + \lambda_t r_{k,t+1} \tag{8}$$

[2]Defining the per-period function in this way implies that the coefficient of relative risk aversion with respect to consumption is equal to θ if $\rho = 1$.

Consumption in the first and second period of life are respectively given by:

$$c_{1,t} + s_t = w_t - \tau_t \tag{9}$$

$$c_{2,t+1} = (1 + r_{t+1})\, s_t + z_{t+1} w_{t+1} \tag{10}$$

where τ_t are lump-sum taxes to finance the interest obligations on the government debt. Maximising life-time utility with respect to consumption ($c_{1,t}$ and $c_{2,t+1}$) and the portfolio allocation (λ_t) subject to the budget constraints gives the following Euler condition:

$$c_{1,t}^{-\theta} = \beta\, \mathrm{E}_t \left[(1 + r_{j,t+1})\, c_{2,t+1}^{-\rho} v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] \tag{11}$$

for $j = b, k$ and with $\phi \equiv \theta - \gamma(1-\rho)$.

The first-order condition with respect to labour supply (z_{t+1}) differs between flexible and inflexible retirement.[3] In the first case, the optimality condition is:

$$\left(\frac{c_{2,t+1}}{1 - z_{t+1}} \right)^{\rho} = \frac{w_{t+1}}{\eta} \tag{12}$$

with $\eta \equiv \gamma/(1-\gamma)$. In the optimum, the marginal rate of substitution between leisure and consumption is equal to the wage rate. Since agents can freely adjust labour supply in period $t+1$, this decision is conditional on the shocks that affect consumption and the wage rate in that period, i.e., $\omega_{A,t+1}$ and $\omega_{\delta,t+1}$. With inflexible retirement, though, the first-order condition is:

$$\mathrm{E}_t \left[\eta\, (1 - z_{t+1})^{-\rho}\, v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] = \mathrm{E}_t \left[w_{t+1} c_{2,t+1}^{-\rho} v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] \tag{13}$$

Since agents are not able to condition the retirement decision at the state of the economy in $t+1$, they have to form expectations. Obviously, z_{t+1} is known at time t.

2.3 Government

The government debt per young worker, b_{t+1}, is equal to the amount of debt in the previous period plus the interest obligations on the outstanding debt minus the collected tax receipts. That is,

$$b_{t+1} = (1 + r_{b,t})\, b_t - \tau_t \tag{14}$$

[3] Throughout the analysis, z_{t+1} indicates labour supply in the second period of life. Under fixed retirement, however, z_{t+1} is chosen in the first period and therefore known at time t.

The government can accumulate debt for a certain amount of time, but at some point in time it has to raise additional taxes in order to keep debt per young worker constant, i.e., $b_{t+1} = b_t = b$. These lump-sum taxes are denoted by τ and are equal to:

$$\tau_t = r_{b,t} b \tag{15}$$

Like the capital stock and labour supply (in case of fixed retirement), the bond return $r_{b,t}$ is a predetermined variable: it denotes the interest that is paid at time t on the government debt that is issued one period before, in $t-1$.

2.4 Equilibrium

The capital market (and the goods market as well) is in equilibrium when savings at time t finance the capital stock and the government debt in the next period:

$$s_t = k_{t+1} + b_{t+1} \tag{16}$$

Moreover, the portfolio allocation has to be such that the right amount of private savings goes to the capital stock and the government debt:

$$\lambda_t s_t = k_{t+1} \tag{17}$$

This implies that there are two equilibrium conditions and k_{t+1} and $r_{b,t+1}$ adjust to make sure that these equilibrium conditions are satisfied.

The complete model is summarized in Table 1. To construct equation (T1.1) we have substituted equations (15) and (16) in equation (9). Equation (T1.2) is the result of inserting the portfolio rate of return (8) and the equilibrium conditions (16) and (17) into equation (10). The remaining equations, equation (T1.4)-(T1.7b), just repeat equation (11) (for $j = k$ and $j = b$) and equations (2), (3), (12) and (13).

3 Solving the model

There are various ways to solve this model. One way is to solve the model numerically using dynamic programming methods or using perturbation methods around the deterministic steady state (see, for instance, Collard and Juillard, 2001 or Schmitt-Grohé and Uribe, 2004). Another possibility is to approximate the model using log-linearization around the steady state. The latter gives a bit more insight into the working of the model,

Table 1: Summary of model equations

$$w_t - c_{1,t} - r_{b,t} b = b + k_{t+1} \tag{T1.1}$$

$$c_{2,t} = (1 + r_{b,t})\, b + (1 + r_{k,t})\, k_t + z_t w_t \tag{T1.2}$$

$$c_{1,t}^{-\theta} = \beta\, \mathrm{E}_t \left[(1 + r_{k,t+1})\, c_{2,t+1}^{-\rho} v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] \tag{T1.3}$$

$$c_{1,t}^{-\theta} = \beta\, (1 + r_{b,t+1})\, \mathrm{E}_t \left[c_{2,t+1}^{-\rho} v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] \tag{T1.4}$$

$$w_t = (1 - \alpha) A_t k_t^{\alpha}\, (1 + z_t)^{-\alpha} \tag{T1.5}$$

$$r_{k,t} + \delta_t = \alpha A_t k_t^{\alpha-1}\, (1 + z_t)^{1-\alpha} \tag{T1.6}$$

$$\left(\frac{c_{2,t+1}}{1 - z_{t+1}} \right)^{\rho} = \frac{w_{t+1}}{\eta} \tag{T1.7a}$$

$$\mathrm{E}_t \left[w_{t+1} c_{2,t+1}^{-\rho} v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] = \mathrm{E}_t \left[\eta\, (1 - z_{t+1})^{-\rho}\, v(c_{2,t+1}, z_{t+1})^{\rho-\phi} \right] \tag{T1.7b}$$

and it is the road we will take in this paper. It should be understood that log-linearization is a small-shock approximation or an approximation to shocks with bounded support (Samuelson, 1970). Despite these limitations of log-linear approximations, this method clearly helps to explore the most important economic factors that affect the interaction between retirement behaviour and portfolio choice. As such, it provides a useful starting point for further qualitative explorations with higher-order numerical techniques.[4]

3.1 The steady state

A linearization around a deterministic steady state is sufficient for understanding macroeconomic dynamics, but it is not necessarily sufficient for an economic analysis involving uncertainty, such as questions about precautionary savings and asset-pricing issues. Following Juillard and Kamenik (2005), Beetsma and Bovenberg (2009) and Bovenberg and Uhlig (2008), we therefore use the concept of a *stochastic* steady state. This concept is defined as a situation in which each period shocks are equal to their expectations but agents are not aware of this (i.e., conditional variances are not zero). This point is solved from a nonlinear system, and hence the solution does not generally correspond to the expected values of the variables involved.[5]

The complete system of steady-state equations is described in Table 2. Variables without time index refer to steady-state values. Notice that equations (T2.1), (T2.2), (T2.5), (T2.6) and (T2.7a) have exactly the same form as the original model equations of Table 1. The remaining expectational equations, i.e., equations (T2.3), (T2.4) and (T2.7b), are derived using second-order Taylor approximations of the original first-order conditions.[6] The use of a stochastic steady state implies that risk terms $\sigma^2_{r_k-v}$, σ^2_v, $\sigma^2_{w-c_2}$ and $\sigma^2_{c_2}$ show up in the first-order conditions reflecting a precautionary motive for saving and postponing retirement. These conditional (co)variances are defined as:

$$\sigma^2_{r_k-v} \equiv \mathrm{Var}_t\left[\log\left(1+r_{k,t+1}\right) - \phi\log c_{2,t+1} + \eta(\rho-\phi)\log(1-z_{t+1})\right] \tag{18}$$

$$\sigma^2_{v} \equiv \mathrm{Var}_t\left[-\phi\log c_{2,t+1} + \eta(\rho-\phi)\log(1-z_{t+1})\right] \tag{19}$$

$$\sigma^2_{w-c_2} \equiv \mathrm{Var}_t\left(\log w_{t+1} - \phi\log c_{2,t+1}\right) \tag{20}$$

$$\sigma^2_{c_2} \equiv \mathrm{Var}_t\left[(\rho-\phi)\log c_{2,t+1}\right] \tag{21}$$

[4] We also checked our results with higher order approximations using Dynare++. Although quantitatively the results give some small differences, the qualitative observations are exactly the same.

[5] Since the solution is not necessarily equal to expected values of the variables, Beetsma and Bovenberg (2009) label this solution as the *median solution*. We prefer to use the term stochastic steady state to indicate that the steady state is adjusted for risk.

[6] See Appendix A.1 for more details. See also Viceira (2001).

Table 2: The steady-state equations

$$w - c_1 - r_b b = b + k \tag{T2.1}$$

$$c_2 = (1 + r_b)\, b + (1 + r_k)\, k + zw \tag{T2.2}$$

$$c_1^{-\theta} = \beta\,(1 + r_k)\, c_2^{-\phi}(1 - z)^{\eta(\rho-\phi)} \exp\left(\frac{1}{2}\sigma^2_{r_k - v}\right) \tag{T2.3}$$

$$c_1^{-\theta} = \beta\,(1 + r_b)\, c_2^{-\phi}(1 - z)^{\eta(\rho-\phi)} \exp\left(\frac{1}{2}\sigma^2_{v}\right) \tag{T2.4}$$

$$w = (1 - \alpha) A k^{\alpha}(1 - z)^{-\alpha} \tag{T2.5}$$

$$r_k + \delta = \alpha A k^{\alpha - 1}(1 + z)^{1-\alpha} \tag{T2.6}$$

$$\left(\frac{c_2}{1 - z}\right)^{\rho} = \frac{w}{\eta} \tag{T2.7a}$$

$$\left(\frac{c_2}{1 - z}\right)^{\rho} = \frac{w}{\eta} \exp\left[\frac{1}{2}\left(\sigma^2_{w - c_2} - \sigma^2_{c2}\right)\right] \tag{T2.7b}$$

At this point, we implicitly assume that these variances are constant over time. This will be justified in the next subsection, when solving for the linear recursive law of motion of the log-linearized system.

In general, the system in Table 2 can not be solved analytically. Only for a particular situation we are able to obtain explicit solutions, namely if: i) life-time utility is log-linear in consumption and leisure ($\theta = \rho = 1$); ii) there is full depreciation ($\delta = 1$) and iii) all conditional covariances are perceived to be zero (deterministic steady state). In that case, we obtain the following analytical expressions for retirement z and the capital-labour ratio $k/(1+z)$:[7]

$$z(\lambda) = \frac{\lambda(1-\alpha) - \alpha\eta}{\lambda(1+\eta-\alpha) + (1-\lambda)\alpha\eta} \tag{22}$$

$$\frac{k}{1+z}(\lambda) = \left[\frac{\alpha\beta A(1+\eta+\alpha)\lambda - 2\alpha^2\beta A}{(1-\alpha)\lambda + \alpha\beta(2+\eta)\lambda + 2\alpha}\right]^{\frac{1}{1-\alpha}} \tag{23}$$

Notice from these expressions that both labour supply and the capital-labour ratio positively depend on the portfolio share λ invested in firm stocks: if λ decreases, for example because of a higher government debt, this leads to a crowding out of firm stocks which reduces the capital-labour ratio. In general equilibrium, a lower capital-labour ratio reduces the wage rate and, hence, labour supply incentives. Simulations confirm that this property of the model also holds under more general assumptions for which analytical results are not available. Given a solution to equations (22) and (23), all other steady-state variables can be calculated.

3.2 The log-linearized model

In the usual situation of a non-stochastic steady state, this steady state can be computed separately from the recursive laws of motion. With a stochastic steady state, though, this procedure does no longer apply. In this case, deriving the recursive laws involves the calculation of a fixed point: note from equations (T2.3), (T2.4) and (T2.7b) that the steady state requires knowledge of the conditional variances, which can be calculated, given the log-linear recursive law of motion. But the latter is a solution to a system of equations of which the coefficients depend on the steady state. Hence, we are forced to simultaneously solve for the steady state and the log-linear recursive laws of motion. Throughout the paper, we use the following notation for log-linearized variables: $\hat{x}_t \equiv \log x_t - \log x$. The complete log-linearized model is reported in Table 3.

[7] See Appendix A.2 for the formal derivation.

Table 3: The log-linearized model

$$w\hat{w}_t - c_1\hat{c}_{1,t} = k\hat{k}_{t+1} + r_b b\hat{r}_{b,t} \tag{T3.1}$$

$$c_2\hat{c}_{2,t} = r_k k\hat{r}_{k,t} + (1+r_k)\,k\hat{k}_t + r_b b\hat{r}_{b,t} + zw\,(\hat{z}_t + \hat{w}_t) \tag{T3.2}$$

$$\phi\,\mathrm{E}_t\,\hat{c}_{2,t+1} - \theta\hat{c}_{1,t} = \frac{r_k}{1+r_k}\,\mathrm{E}_t\,\hat{r}_{k,t+1} - \eta(\rho-\phi)\frac{z}{1-z}\,\mathrm{E}_t\,\hat{z}_{t+1} \tag{T3.3}$$

$$\phi\,\mathrm{E}_t\,\hat{c}_{2,t+1} - \theta\hat{c}_{1,t} = \frac{r_b}{1+r_b}\hat{r}_{b,t+1} - \eta(\rho-\phi)\frac{z}{1-z}\,\mathrm{E}_t\,\hat{z}_{t+1} \tag{T3.4}$$

$$\hat{w}_t = \alpha\hat{k}_t - \alpha\frac{z}{1+z}\hat{z}_t + \omega_{A,t} \tag{T3.5}$$

$$\hat{r}_{k,t} + \frac{\delta}{r_k}\hat{\delta}_t = \frac{r_k+\delta}{r_k}\left[(1-\alpha)\frac{z}{1+z}\hat{z}_t - (1-\alpha)\hat{k}_t + \omega_{A,t}\right] \tag{T3.6}$$

$$\hat{z}_{t+1} = \frac{1-z}{\rho z}\hat{w}_{t+1} - \frac{1-z}{z}\hat{c}_{2,t+1} \tag{T3.7a}$$

$$\hat{z}_{t+1} = \frac{1-z}{\rho z}\,\mathrm{E}_t\,\hat{w}_{t+1} - \frac{1-z}{z}\,\mathrm{E}_t\,\hat{c}_{2,t+1} \tag{T3.7b}$$

Solving for the steady state and the log-linearized equilibrium laws involves a three-step procedure. The first step is to write the log-linearized endogenous variables as function of the endogenous and exogenous state variables. Our model contains two exogenous state variables, productivity shocks ($\omega_{A,t}$) and depreciation shocks (ω_δ) and one endogenous state variable, which is the capital stock ($\hat{k}_t$). Recall that the return on government bonds ($\hat{r}_{b,t}$) and labour supply in case of retirement inflexibility ($\hat{z}_t$) are predetermined variables at time t. It turns out, however, that both variables are proportional to the capital stock so that they can be eliminated from the state space.[8]

The proportional (and negative) relation between the return on bonds and the capital stock follows from capital-market equilibrium: a higher capital stock combined with a constant level of government debt has to result in a more aggressive asset portfolio. To make this happen, the risk-free return on bonds will fall. The proportional relation between labour supply and the capital stock in case of retirement inflexibility can either be positive or negative, depending on the relative strength of income and substitution effects: a higher next-period capital stock leads to higher future wage expectations. Hence, rational agents, who plan to retire before shocks are revealed under retirement inflexibility, will postpone retirement if the substitution effect dominates and will advance retirement if the income effect dominates.

Accordingly, the capital stock is the only endogenous state variable in the model. For any endogenous variable $\hat{x}_t$ we are looking for the following recursive equilibrium law:

$$\hat{x}_t = \pi_{x,k}\hat{k}_t + \pi_{x,A}\omega_{A,t} + \pi_{x,\delta}\omega_{\delta,t} \tag{24}$$

where $\pi_{x,k}$ is the partial elasticity of $\hat{x}_t$ with respect to $\hat{k}_t$, $\pi_{x,A}$ is the partial elasticity of $\hat{x}_t$ with respect to $\omega_{A,t}$ and $\pi_{x,\delta}$ is the partial elasticity of $\hat{x}_t$ with respect to $\omega_{\delta,t}$.[9]

As a second step, we use the derived recursive law to write the conditional variances in terms of the steady-state values and the exogenous shock terms. Then we obtain for the variance terms of the Euler equations:

$$\sigma^2_{r_k - v} \equiv \sum_{i=A,\delta} \left[\frac{r_k}{1+r_k}\pi_{r_k,i} - \phi\pi_{c_2,i} - \frac{\eta(\rho-\phi)z}{1-z}\pi_{z,i} \right]^2 \sigma_i^2 \tag{25}$$

$$\sigma^2_{v} \equiv \sum_{i=A,\delta} \left[-\phi\pi_{c_2,i} - \frac{\eta(\rho-\phi)z}{1-z}\pi_{z,i} \right]^2 \sigma_i^2 \tag{26}$$

[8] See Appendix B, equations (A.19) and (A.21), for a formal proof of this statement.

[9] The partial elasticities of the endogenous variables are derived in Appendix B.1 (flexible retirement) and Appendix B.2 (fixed retirement).

$$\sigma^2_{w-c_2} \equiv \sum_{i=A,\delta} \left(\pi_{w,i} - \phi\pi_{c_2,i}\right)^2 \sigma_i^2 \tag{27}$$

$$\sigma^2_{c_2} \equiv \sum_{i=A,\delta} \left[(\rho - \phi)\pi_{c_2,i}\right]^2 \sigma_i^2 \tag{28}$$

Note that these variances are indeed constant over time, as assumed in the previous subsection. Equations (25) and (26) apply to the flexible retirement setting as well as to the inflexible retirement setting, but the partial elasticities differ in both cases. Equations (27) and (28) only apply to the inflexible retirement setting.

In the final step, we numerically solve for the steady-state variables, given the derived expressions for the conditional variances. In case of retirement flexibility, this boils down to solving equations (T2.1)-(T2.7a), equation (25) and equation (26). For retirement inflexibility, the complete system of equations is described by equations (T2.1)-(T2.6), (T2.7b) and (25)-(28). Once solved for the steady state, the computed formulas in Appendix B.1 (for flexible retirement) and Appendix B.2 (for flexible retirement) retrieve the partial derivatives, and hence, the linear recursive system.

4 Retirement as hedge: some analytics

The current literature on retirement flexibility and portfolio choice only focuses on partial-equilibrium models and mainly sticks to capital-market risks. The main result that can be derived from this literature is that flexibility in the retirement decision increases the fraction of wealth invested in equity.[10] Viewed in this way, labour supply flexibility creates a kind of insurance against adverse investment outcomes. In this section, we illustrate this benchmark result in the context of our model. With reference to the literature, we take a partial-equilibrium perspective (factor prices are exogenous) and assume that there is only capital-market risk implying that wages are non-stochastic. To keep the analysis as simple as possible, we impose that expected life-time utility is log-linear in first-period consumption, second-period consumption and leisure (i.e., $\rho = \theta = 1$).

To derive an explicit solution for the portfolio choice λ_t, we follow the approach of Hansen and Singleton (1983) and Campbell and Viceira (2002) and assume that the joint distribution of consumption and asset returns is lognormal. Then the optimal solution for portfolio choice in case of flexible retirement is given by (see Appendix C.1):

$$\lambda_t^F = \left[1 + \frac{w_{t+1}}{(1 + r_{b,t+1})\, s_t}\right] \frac{\log \mathrm{E}_t\left(1 + r_{k,t+1}\right) - \log\left(1 + r_{b,t+1}\right)}{\mathrm{Var}_t \log\left(1 + r_{k,t+1}\right)} \tag{29}$$

[10] See, e.g., Bodie et al. (1992), Choi et al. (2008), Choi and Shim (2006) or Farhi and Panageas (2007).

The optimal investment share in the risky asset is increasing in the expected excess return of the risky asset and decreasing in its variance. In case of inflexible retirement, the optimal equity share equals (see Appendix C.2):

$$\lambda_t^I = \left[1 + \frac{w_{t+1} z_{t+1}}{(1 + r_{b,t+1})\, s_t}\right] \frac{\log \mathrm{E}_t\,(1 + r_{k,t+1}) - \log\,(1 + r_{b,t+1})}{\mathrm{Var}_t \log\,(1 + r_{k,t+1})} \tag{30}$$

Note that equation (29) and equation (30) are identical except for one factor: λ^F contains maximum potential human capital while λ^I contains actual labour income which is scaled by $z_{t+1} < 1$.[11] Hence, it is straightforward to derive the following result:

Result 1. *The investment allocation to the risky asset is larger in the case of flexible retirement compared to the inflexible retirement case, i.e.,* $\lambda_t^F > \lambda_t^I$.

Result 1 is well-known from the literature, and was first derived by Bodie et al. (1992).[12] If agents have the possibility to postpone retirement after an adverse shock, they can afford to take more investment risk during working life. As shown by equations (29) and (30), this higher risk taking stems from a wealth effect. The demand for the risky asset depends positively on the amount of human wealth of an individual. With flexible retirement, the individual has in effect a larger store of human capital upon which to draw. Since human capital is risk free (at least until now), the individual rebalances his total wealth holdings by investing a larger share of financial wealth in the risky asset. By contrast, with fixed retirement an individual has a smaller amount of potential human capital from which to invest and therefore requires less rebalancing.

Obviously, these differences in portfolio allocation have consequences for the retirement decision. With flexible labour supply, the optimal solution for retirement is equal to (see again Appendix C.1):

$$z_{t+1}^F = 1 - \frac{\eta\beta\,(1 + r_{T,t+1})}{1 + \beta(1 + \eta)} \left(\frac{w_t - \tau_t}{w_{t+1}} + \frac{1}{1 + r_{b,t+1}}\right) \tag{31}$$

with,

$$r_{T,t+1} \equiv (1 - a_t) r_{b,t+1} + a_t r_{k,t+1} \tag{32}$$

$$a_t \equiv \frac{\lambda_t s_t}{s_t + \frac{w_{t+1}}{1 + r_{b,t+1}}} \tag{33}$$

[11] In principle, private savings may not be equal in the flexible and fixed retirement case. However, in Appendix C we show that $s_t^F = s_t^I$.

[12] Bodie et al. (1992) show that this result also holds for more general utility functions.

Note that a_t is the fraction of an individual's total wealth (financial wealth plus human wealth) invested in the risky asset. Hence, $r_{T,t+1}$ is the effective return on the individual's total portfolio when human wealth (i.e., the discounted value of future labour income) is also taken into account. In case of a positive equity shock, i.e., r_T is high, agents will retire earlier due to a positive wealth effect, and vice versa. With inflexible retirement, the optimal retirement decision equals (see again Appendix C.2):

$$z_{t+1}^I = 1 - \frac{\eta\beta\,(1+r_{b,t+1})}{1+\beta(1+\eta)}\left(\frac{w_t-\tau_t}{w_{t+1}}+\frac{1}{1+r_{b,t+1}}\right) \tag{34}$$

Note that the risk-free return $r_{b,t+1}$ now enters the retirement function rather than the stochastic return $r_{T,t+1}$. Accordingly, it is possible to derive the following result:

Result 2. *The expected retirement age in the flexible retirement case is lower than in the inflexible case, i.e.,* $\mathrm{E}_t\, z_{t+1}^F < z_{t+1}^I$.

Proof. Using the optimal solution for s_t (derived in Appendix C), it follows from equation (29) that $\lambda_t s_t > 0$. Using equation (33), this implies that $a_t > 0$ and, hence, $\mathrm{E}_t\, r_{T,t+1} > r_{b,t+1}$. □

In summary, when people can adjust their retirement decision, they will invest more in the risky asset. Since the risky asset has a higher expected return, these people can on average afford to retire earlier.

5 Quantitative results

This section explores the quantitative properties of the model and numerically calculates the steady state and the reaction of the various variables to productivity and depreciation shocks. We first use the model to gain insight in the partial-equilibrium effects of retirement (in)flexibility. Then we turn to the general-equilibrium effects and relate these to the partial-equilibrium results.

5.1 Parameterization

In order to quantify the interaction between portfolio choice and retirement, we first have to parameterize the model. We normalize the average productivity parameter at $A = 1$. The capital share in the Cobb-Douglas production function is taken to be $\alpha = 0.3$, as in Krueger and Kubler (2006) and Olovsson (2010). We set δ, the average depreciation rate,

Table 4: Benchmark parameterization

Parameter	β	γ	ρ	θ	α	A	δ	b	σ_A	σ_δ
Values	0.65	0.5	1	2	0.3	1	0.75	0.015	0.31	1.31

to 0.75. Assuming that one model period lasts about 30 years, this corresponds with a depreciation rate of 5 percent per year, like in Olovsson (2010). We choose as benchmark an intertemporal elasticity of substitution of one half, i.e., $\theta = 2$, and an intratemporal substitution of $\rho = 1$. An intertemporal elasticity of substitution of one half lies well within the range of available estimates (see e.g. Attanasio and Weber, 1995 or Blundell et al., 1994) and is commonly used in the macro and public finance literature (it implies a coefficient of relative risk aversion of 2). We choose as time discount factor $\beta = 0.65$, or a time discount rate of 1.4 percent per year, as in Krueger and Kubler (2006). The leisure parameter is set at $\gamma = 0.5$ and the supply of government debt is set at $b = 0.015$, a combination which provides plausible values for the retirement age and the risk-free return on government bonds (see below).

Since productivity risk directly affects all factor prices in the economy (wages and asset returns) and depreciation risk only influences capital returns, the two risk factors certainly have a different effect on retirement and portfolio decisions. We will therefore analyse the model for depreciation and productivity risk separately. In order to make the results comparable, we calibrate the standard deviation of the exogenous shock (i.e., σ_A in case of productivity risk and σ_δ in case of depreciation risk) in such a way that the annualized standard deviation of the return on capital is the same in both cases and equal to 8.2 percent.[13] This leads to $\sigma_A = 0.31$ and $\sigma_\delta = 1.31$. All parameters used in the benchmark model are summarized in Table 4.

5.2 Partial equilibrium

For flexible labour supply, the partial-equilibrium solution is determined by equations (18) and (19), equations (T2.1)-(T2.4) and equation (T2.7a). In case of fixed labour supply, we have to solve for equations (18), (19), (20), (21), equations (T2.1)-(T2.4) and equation (T2.7b).

By definition, in the partial-equilibrium model factor prices are exogenous and only influenced by the exogenous shock terms $\omega_{A,t}$ and $\omega_{\delta,t}$. The log-linearized equations for

[13]Here we follow Campbell and Viceira (2005) who show that returns on stocks are significantly less volatile when the investment horizon is long.

Table 5: Steady state of partial equilibrium models

	Depreciation risk		Productivity risk	
	Fixed	Flexible	Fixed	Flexible
c_1/y	38.47	38.34	38.22	38.19
c_2/y	50.88	49.56	49.50	49.62
s/w	31.28	31.50	32.04	32.09
z	20.86	17.13	16.44	16.97
λ	60.59	78.16	97.21	91.09

Note: all figures are expressed in percentages.

wages and capital returns are thus:

$$\hat{w}_t = \omega_{A,t} \tag{35}$$

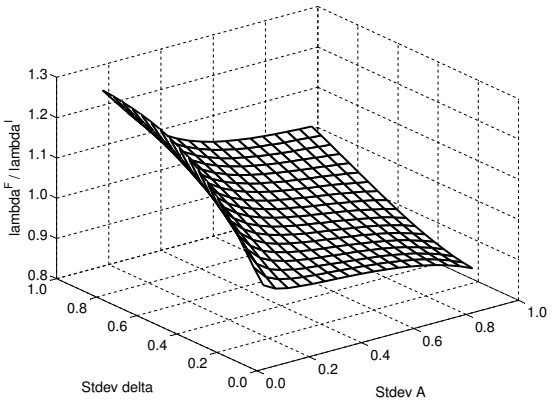
$$\hat{r}_{k,t} = \frac{r_k + \delta}{r_k}\omega_{A,t} - \frac{\delta}{r_k}\omega_{\delta,t} \tag{36}$$

The partial elasticities of the wage rate and the return on capital with respect to productivity and depreciation shocks (i.e., $\pi_{w,A}$, $\pi_{w,\delta}$, $\pi_{r_k,A}$ and $\pi_{r_k,\delta}$), as shown in equation (35) and (36), are the same as those derived for the general-equilibrium model with fixed retirement.[14] This makes sense because with fixed labour supply both the capital stock and labour supply are predetermined variables. Conditional on information at time t, the only source of variation in future factor prices comes from the exogenous shocks. Consequently, if the exogenous factor prices are set at the corresponding general-equilibrium values, the partial-equilibrium model gives exactly the same results.

Table 5 compares the steady-state results for fixed and flexible labour supply. The table distinguishes between depreciation and productivity risk. The capital return, the return on bonds and the wage rate are exogenous and obtained from the general-equilibrium model with flexible labour supply. Note that, in case of depreciation risk, our model reproduces the traditional view that retirement flexibility increases risk exposure, the first result analytically derived in the previous section. From equation (35) and (36) we see that wages and capital returns are not correlated when depreciation risk is the only source of uncertainty. A positive depreciation shock (i.e., a negative wealth shock) causes marginal utility from working to increase and, hence, agents increase labour supply (or postpone retirement). Consequently, income effects generate a negative correlation between asset returns and labour income, enabling investors to take greater advantage of

[14] See Appendix B.2.

Figure 1: Reaction of equity share in case of flexible retirement relative to inflexible retirement, when the standard deviations of exogenous shocks are varied



the equity premium. The result of this investment strategy is that retirement flexibility induces agents to retire earlier on average compared to retirement inflexibility, the second result derived in Section 4. Given our parameterization, agents choose to retire after 66.3 years in case of inflexible retirement while they retire on average after 65.1 years in case of flexible retirement, a difference of about 14 months.[15]

If productivity risk is the sole risk factor, however, the results will turn around. In that case, retirement flexibility may instead be used to amplify the productivity shocks absorbed into consumption, leading to less risk exposure and a higher retirement age compared to fixed retirement. The reason is that productivity shocks do not only induce an income effect in labour supply but also a substitution effect which works in the opposite direction. This substitution effect exacerbates the positive correlation between labour income and capital returns, making equity investment relative unattractive under retirement flexibility. When productivity goes down, both the return on capital and the wage rate decrease. When people can freely adjust retirement, they will respond to this lower wage rate by reducing labour supply, which decreases labour income even further. Hence, under retirement flexibility labour supply behaviour is subject to procyclical pressure which reduces the risk bearing capacity of consumers. As a result, people are forced to work longer on average. Given our parameterization, this additional work span amounts almost 2 months.

Figure 1 shows the change of the relative equity share (i.e., the equity share in case of flexible retirement divided by the equity share in case of inflexible retirement) for different

[15]We assume that each generation lasts 30 years. Life time consists of 30 years of childhood and schooling that are not accounted for, 30 years of full activity and a last period of 30 years the first part of which is devoted to working and lasts $30z$ years. The retirement age is thus $60 + 30z$.

values for σ_A and σ_δ in a three-dimensional mesh. The two standard deviations are varied between 0.1 at the lower end and 0.9 at the upper end. When the retirement decision is flexible in the second period of life, agents invest relatively much in equity if depreciation risk is high and productivity risk low and vice versa.

5.3 General equilibrium

Now we turn to the general-equilibrium solution. Table 6 shows the steady-state results in case of general equilibrium and again distinguishes between depreciation and productivity risk. The first column with numbers shows the results for the deterministic steady state, i.e., when the conditional variances are zero.

Comparing the deterministic steady state with the stochastic steady states illustrates the role of uncertainty in the model. Obviously, if there is no uncertainty, the equity premium (i.e., $r_k - r_b$) is equal to zero since capital investments and government bonds are perfect substitutes. In the stochastic steady state, the equity premia are positive reflecting the higher riskiness of capital investments.[16] Including the risk terms in the optimality conditions introduces a precautionary motive for more savings and later retirement. Note that the saving rate and labour supply are higher in the stochastic steady state than in the deterministic steady state.

In general equilibrium, exactly the same risk features appear as in partial equilibrium but they are now operating through price adjustments rather than quantity adjustments. With exogenous factor prices, we saw that agents invest more in equity under flexible labour supply than under fixed labour supply if depreciation risk is the dominant source of uncertainty. When productivity risk is the dominant source, we found the opposite result, namely that agents invest less in equity under retirement flexibility than under retirement inflexibility. With endogenous factor prices and a fixed supply of government bonds, though, different risk attitudes affect the price of risk taking, i.e., the equity premium. If productivity risk is the sole risk factor, the equity premium is *higher* in case of flexible retirement than in case of inflexible retirement. The intuition for this lower risk appetite under flexible retirement is the same as before: the substitution effect related to labour market flexibility exacerbates the positive correlation between asset returns and labour income which decreases the risk appetite. Hence, people are only willing to invest in the domestic capital stock if they receive a higher expected compensation. If there is only depreciation risk, however, the insurance mechanism related to the income effect

[16] Note that the reported risk premia are on the low side, which is a manifestation of the equity premium puzzle.

Table 6: Steady state of general equilibrium models

	No risk	Depreciation risk		Productivity risk	
		Fixed	Flexible	Fixed	Flexible
c_1/y	38.32	37.43	38.41	38.31	38.17
c_2/y	50.10	51.41	49.62	49.36	49.52
s/w	30.48	30.62	31.39	32.05	32.11
$r_k - r_b$	0.00	0.52	0.32	0.32	0.37
r_b	2.62	2.21	2.20	2.11	2.07
z	16.57	21.21	17.04	16.67	17.14
k/y	15.44	14.88	15.96	16.44	16.41
λ	84.34	84.16	85.01	85.50	85.52

Note: the equity premium and the return on government debt are annualized figures. All figures are expressed in percentages.

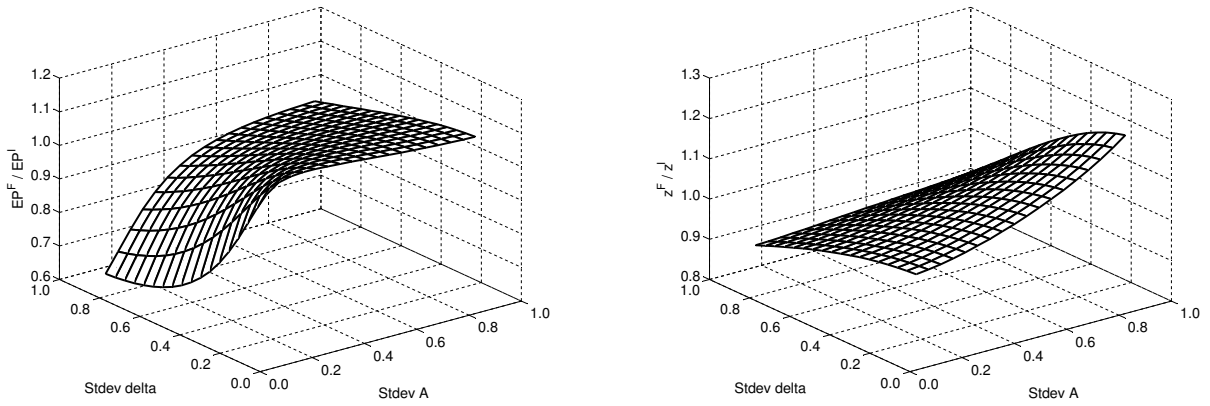
dominates, resulting in a *lower* equity premium under labour market flexibility.

Like in the partial-equilibrium model, steady-state labour supply is lower with flexible retirement than with inflexible retirement if there is only depreciation risk. In the former case, people on average choose to retire after 65.1 years while in the latter case they retire after 66.4 years, a difference of about 15 months. When agents have no retirement flexibility and only face depreciation risk, labour supply is an attractive way to finance future consumption compared to private savings, because wages are not uncertain while the proceeds of savings are uncertain. On the contrary, with retirement flexibility equity savings are attractive because people will probably earn the equity premium while they always have the option to postpone retirement if things go wrong. Hence, compared to the inflexible setting, agents save more and a higher fraction of these savings is allocated to firm equity. Since the supply of government debt is given in general equilibrium, the equity premium has to decline to make sure that enough savings are allocated to this debt. It turns out that the wealth effect (more savings) dominates the price effect (lower equity premium), resulting in lower labour supply under retirement flexibility.

If there is only productivity risk, instead, retirement flexibility is less interesting from an insurance perspective because capital returns are low in states in which wages are also low. Therefore, agents have a relative high demand for risk-free bonds which drives down the interest rate on government debt. This negative wealth effect implies that agents on average retire about 2 months later with flexible labour supply.

Figure 2 shows the dependence of portfolio and retirement decisions on the two risk

Figure 2: Reaction of equity premium and labour supply in case of flexible retirement relative to inflexible retirement, when the standard deviations of exogenous shocks are varied



(a) Equity premium

(b) Labour supply

factors in a more general way. These figures compare the equity premium (left panel) and labour supply (right panel) in case of retirement flexibility with those in case of retirement inflexibility. If depreciation risk is high and productivity risk low, the risk premium is lower under flexible retirement, reflecting the self-insurance role of voluntary retirement. When productivity risk becomes more important, the equity premium increases and ultimately passes the levels of the fixed retirement setting. A comparable pattern emerges for labour supply behaviour. For higher degrees of productivity risk, the hedging effect of retirement flexibility decreases which leads to a higher demand for risk-free government bonds and, given the fixed level of government debt, to lower risk-free interest rates. This negative wealth effect induces agents to postpone retirement.

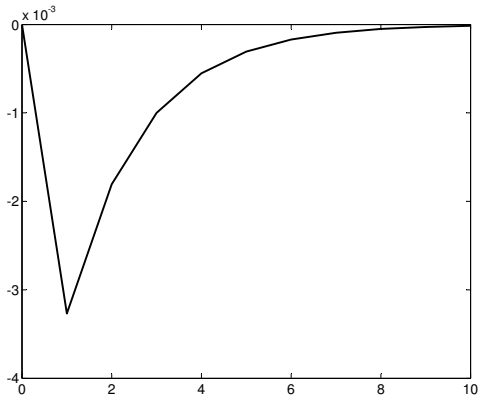
It should be stressed that from a welfare perspective flexibility is always preferable to inflexibility. With retirement flexibility, expected life-time utility is unambiguously higher, both in case of depreciation risk and productivity risk.[17] This result makes sense because the model does not include any distortion or externality.

5.4 Dynamics

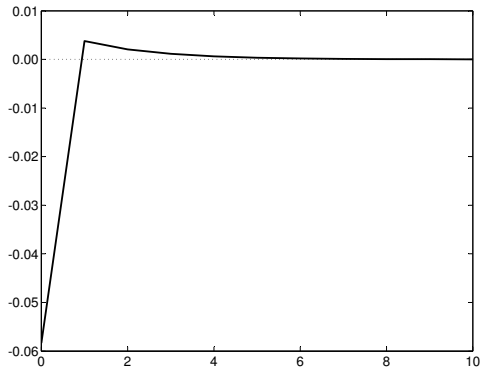
The different roles in the interaction between retirement flexibility and portfolio allocation played by productivity and depreciation shocks can best be illustrated using impulse

[17]By simulating the derived recursive laws, we have calculated the unconditional means of most important model variables. It turns out that the unconditional mean of life-time utility in case of flexible retirement is always higher than that in case of inflexible retirement.

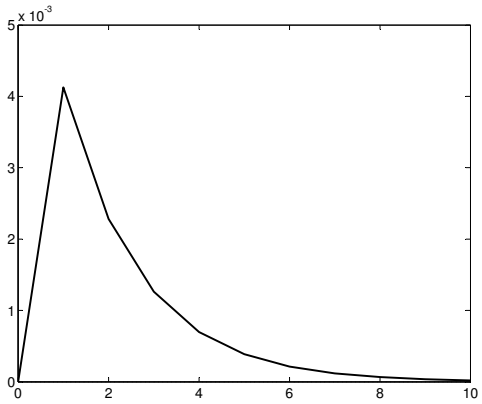
Figure 3: Impulse responses to a positive 10 percent depreciation shock, given the benchmark parameterization



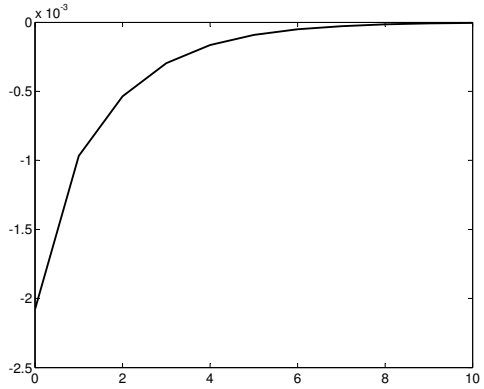
(a) Capital stock



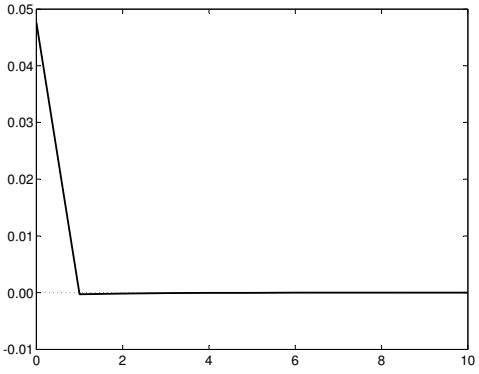
(b) Capital return



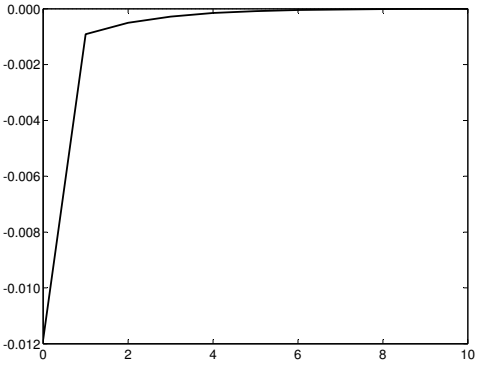
(c) Bond return



(d) Wage rate

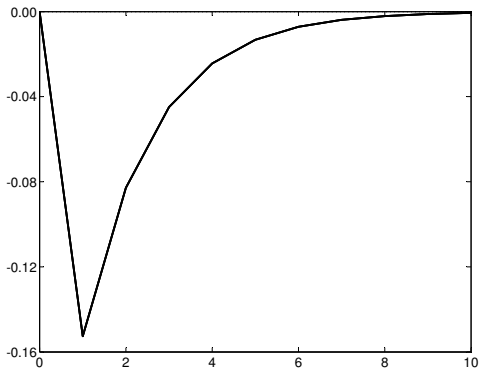


(e) Labour supply

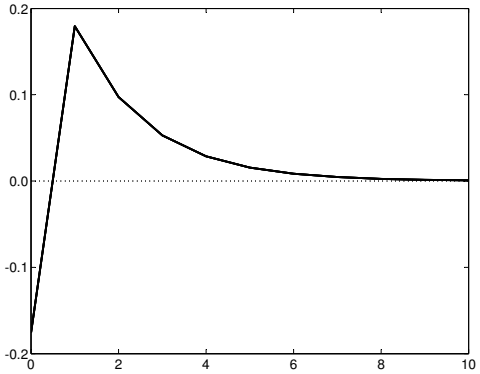


(f) Old-age consumption

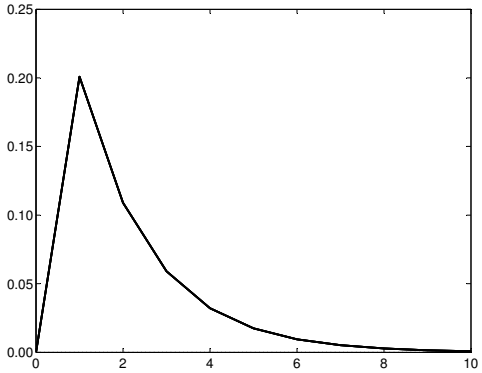
Figure 4: Impulse responses to a negative 10 percent productivity shock, given the benchmark parameterization



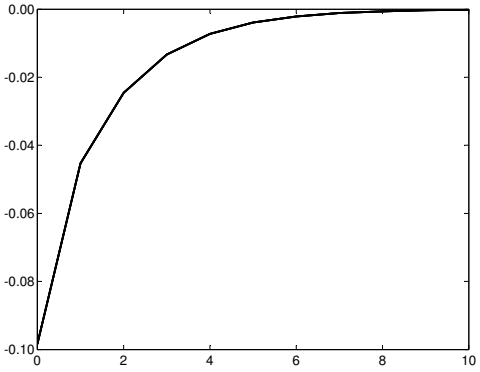
(a) Capital stock



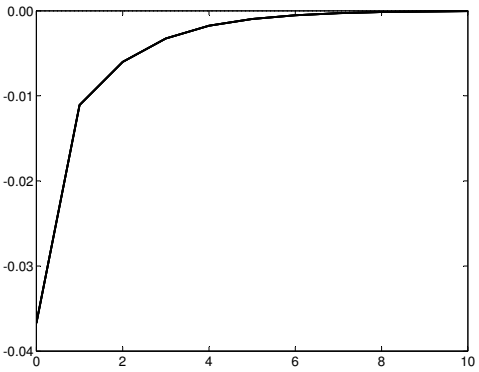
(b) Capital return



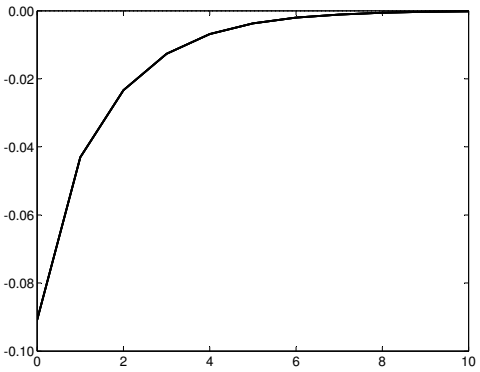
(c) Bond return



(d) Wage rate



(e) Labour supply



(f) Old-age consumption

response functions. Figure 3 shows the response of the capital stock, the return on capital and bonds, the wage rate, labour supply and old-age consumption to a 10 percent positive depreciation shock. These responses are expressed in percent deviation from the steady state. Figure 4 shows the impulse responses for a negative productivity shock of 10 percent.

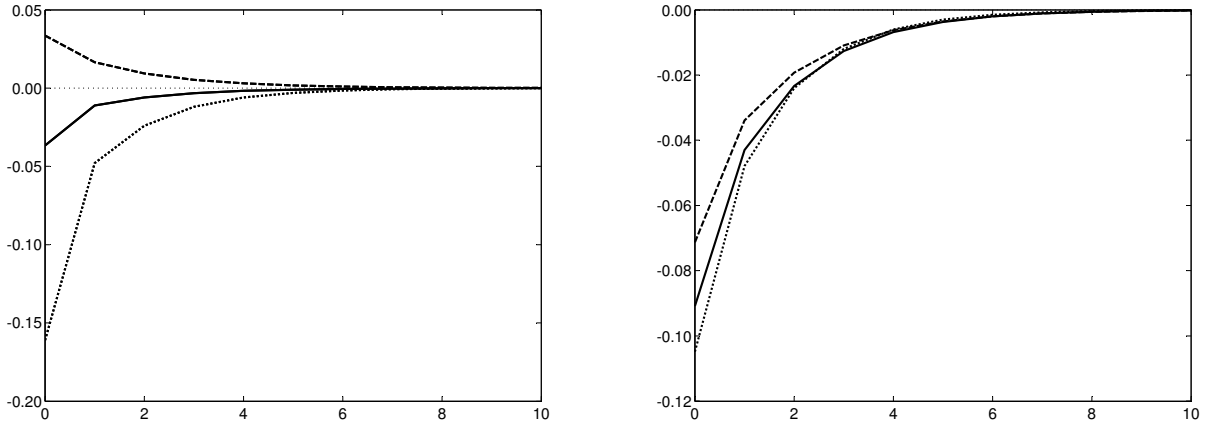
Note first that depreciation shocks lead to relative small responses compared to productivity shocks. After a depreciation shock of 10 percent, the capital return immediately decreases and, due to the income effect, labour supply increases. This negative correlation between the capital return and labour supply moderates consumption volatility and that is why flexibility provides insurance against adverse shocks. At impact, the decline of old-age consumption is small compared to the decline of the capital return. The capital stock is a predetermined variable and falls one period later. This lower level of the capital stock increases its marginal product so that labour supply declines and, hence, wages and consumption gradually return to their pre-shock levels. The return on bonds moves in the opposite direction of the capital stock: a lower capital stock increases its marginal product leading to a higher demand for capital investment and a lower demand for bond investments. As a result, the return on bonds should increase in order to ensure that the fixed supply of government debt will be financed each period.

The economic responses after a productivity shock are much larger. In this case, the decrease in the capital return is even larger than the initial decline in productivity itself. Compared to a depreciation shock, a productivity shock does not only directly affect the return on capital but also the wage rate which falls at impact. This shock induces income and substitution effects in labour supply. Indeed, given the benchmark parameterization, the substitution effect dominates the income effect and that is why labour supply slightly decreases. Hence, productivity shocks result in pro-cyclical labour supply behaviour which exacerbates consumption volatility. Note that the initial decline in old-age consumption is almost as high as the relative decrease in productivity. From an investment point of view, the positive co-movement between capital returns and labour income reduces the appetite for risk taking. Consequently, the equity premium will be relatively higher under retirement flexibility.

5.5 Substitution between consumption and leisure

The previous analysis has shown that the insurance effect of retirement flexibility very much depends on income and substitution effects in labour supply. In our benchmark parameterization, the substitution effect slightly dominates the income effect so that

Figure 5: Impulse responses for a negative 10 percent productivity shock, for $\rho = 0.8$ (dotted line), $\rho = 1$ (solid line) and $\rho = 2$ (dashed line)



(a) Labour supply

(b) Consumption

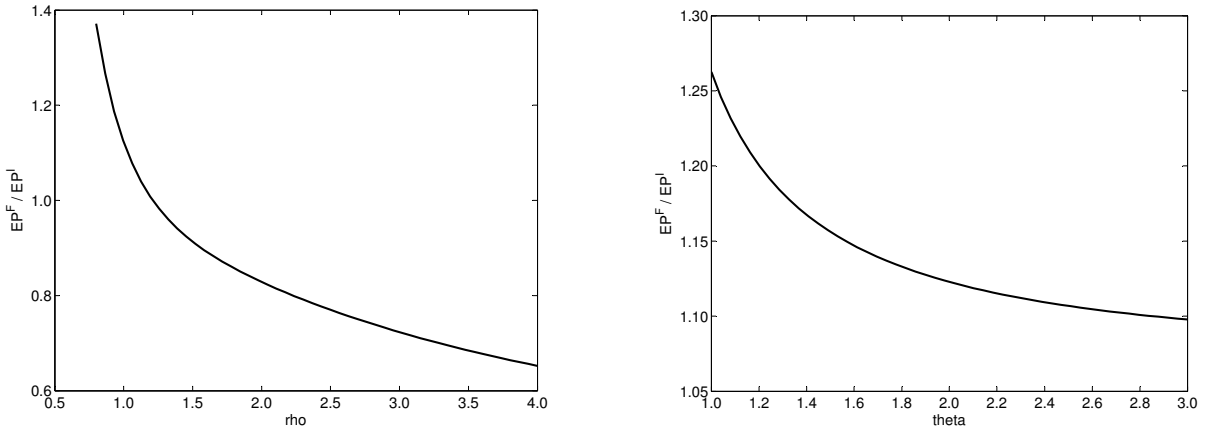
old-age consumption becomes more sensitive to productivity risk in case of retirement flexibility. As a result, agents ask for a higher risk compensation (in general equilibrium) or decrease the equity share in the total asset portfolio (in partial equilibrium).

The relative strength of income and substitution effects is governed by the elasticity of substitution between consumption and leisure (i.e., $1/\rho$). Figure 5 shows the responses of labour supply and consumption to a negative productivity shock of (again) 10 percent for various degrees of substitutability between consumption and leisure. The dotted line is based on an elasticity of substitution of 1.25, the solid line repeats the benchmark case of a unit elasticity and the dashed line is based on an elasticity of substitution of 0.5. Indeed, for a higher (lower) elasticity, the substitution effect becomes relatively more (less) important. In case the elasticity of substitution is 1.25, labour supply actually decreases by more than 15 percent after a drop in productivity of 10 percent. If this elasticity is 0.5, instead, labour supply increases by 3 percent. As one can see, these labour supply responses make old-age consumption more pro-cyclical if the elasticity of substitution is high and vice versa.

When retirement is flexible, the positive comovement of consumption and labour leads to higher equity premia if the elasticity of substitution increases. Figure 6 (left panel) shows the reaction of the equity premium in case of retirement flexibility relative to the equity premium in case of inflexibility for different degrees of substitution between consumption and leisure.[18] For low values of ρ (high elasticity of substitution), the

[18]In Figure 6, it is assumed that productivity risk is the sole risk factor, because substitution effects in labour supply are not relevant in case of depreciation risk.

Figure 6: Reaction of equity portfolio investment in case of flexible retirement relative to inflexible retirement, when the intratemporal elasticity of substitution (left panel) and the coefficient of relative risk aversion (right panel) are varied



(a) Intratemporal substitution

(b) Risk aversion

equity premium under flexible retirement exceeds the equity premium under inflexible retirement. For higher values of ρ (lower elasticity of substitution), the income effect becomes gradually more important and, hence, also the insurance effect of retirement flexibility increases. So when the elasticity of substitution is high, retirement flexibility acts in the direction of resolving the equity risk premium puzzle (Basak, 1999).

The right panel of Figure 6 illustrates the sensitivity of the relative equity premium, now for different degrees of risk aversion (or intertemporal substitution). As one can see, for all values of θ considered, the ratio is decreasing in relative risk aversion but it never falls below unity. This means that, contrary to the elasticity of intratemporal substitution, the coefficient of relative risk aversion does not alter the order of the equity premium: the equity premium is higher with flexible retirement than with fixed retirement.

5.6 Importance of general-equilibrium effects

An interesting question is whether the general-equilibrium effects increase or decrease the risk appetite compared to a partial-equilibrium approach.[19] Existing studies in the field of retirement and portfolio choice only focus on partial-equilibrium models thereby ignoring the potentially important general-equilibrium effects. Our model can be used to isolate the general-equilibrium effects of retirement flexibility and to identify the main factors

[19]Remember that under fixed retirement the partial-equilibrium solution coincides with the general-equilibrium solution. Hence, in this section, the comparison between partial and general equilibrium only points to flexible retirement.

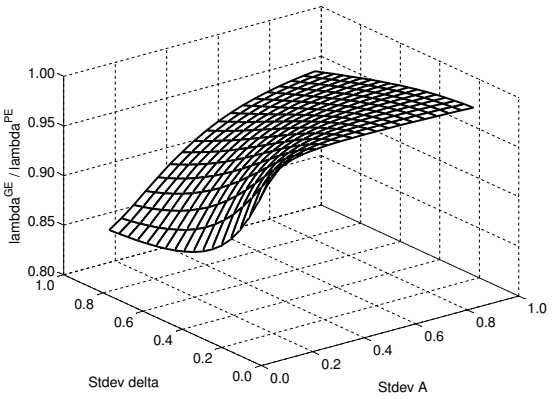
that determine the direction of these effects. As will be discussed, the differences between general-equilibrium and partial-equilibrium results can be reduced to differences in the partial elasticities of the capital return and labour supply with respect to the exogenous shocks (i.e., $\pi_{r_k,A}$, $\pi_{z,A}$, $\pi_{r_k,\delta}$ and $\pi_{z,\delta}$). Recall from equations (25) and (26) that these elasticities determine the conditional variances $\sigma^2_{r_k-v}$ and σ^2_v under flexible retirement.

Figure 7 shows the portfolio share of equity in general equilibrium compared to that in partial equilibrium, again plotted for various degrees of productivity and depreciation risk. In order to make a comparison possible, for each combination of standard deviations, the exogenous factor prices in partial equilibrium are imposed to be the same as the calculated factor prices in general equilibrium. Panel (a) is based on log-linear life-time utility ($\theta = 1$ and $\rho = 1$). On the whole grid of standard deviations, the relative equity exposure is below one meaning that in general equilibrium agents invest less in equity than in partial equilibrium. Note that this difference in risk exposure is particularly large if depreciation risk is high. Since everyone decides to work longer (or to postpone retirement) after an adverse depreciation shock, wages will decline in general equilibrium. Consequently, the positive elasticity of labour supply with respect to depreciation shocks ($\pi_{z,\delta}$) is lower in general equilibrium which makes the insurance of retirement flexibility less effective. Optimizing agents respond to this by lowering their risk exposure. At the same time, the higher supply of labour will also moderate the decline of the capital return in general equilibrium. In other words, the elasticity of the capital return with respect to depreciation shocks ($\pi_{r_k,\delta}$) is less negative than in partial equilibrium. This improves the effectiveness of the insurance and, hence, tends to boost risky investments. With this parameterization, though, the negative effect on risky investments (due to a lower $\pi_{z,\delta}$) dominates the positive effect (due to a less negative $\pi_{r_k,\delta}$).

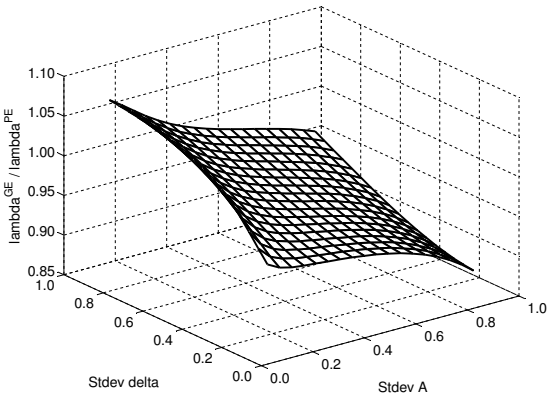
Why is the relative equity share still below unity for higher degrees of productivity risk? As seen before, with an elasticity of substitution equal to one, agents choose to advance retirement after a negative productivity shock (see panel (e) of Figure 4). In other words, the substitution effect dominates the income effect in labour supply (i.e., $\pi_{z,A} > 0$). In general equilibrium, this reduction in labour supply exacerbates the direct fall of the capital return on account of the productivity contraction. Hence, the capital return is more sensitive to productivity risk than in partial equilibrium (i.e., $\pi_{r_k,A}$ higher) which decreases the effectiveness of the hedging effect of retirement flexibility.

If we increase risk aversion (see panel (b) of Figure 7), the insurance effect is still less effective in general equilibrium for higher levels of productivity risk. However, it becomes more effective for lower degrees of productivity risk and higher degrees of depreciation

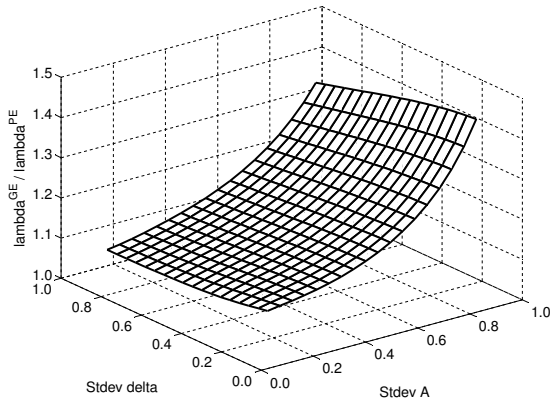
Figure 7: Reaction of equity share in case of general equilibrium relative to partial equilibrium, when the standard deviations of exogenous shocks are varied



(a) $\theta = 1$ and $\rho = 1$



(b) $\theta = 2$ and $\rho = 1$



(c) $\theta = 2$ and $\rho = 2$

risk. If risk aversion is higher, the relatively low sensitivity of the capital return with respect to depreciation risk in general equilibrium (which improves the effectiveness of the insurance effect) now dominates the relatively low response in labour supply (which worsens the effectiveness).

In the previous section, we have seen that the elasticity of substitution between consumption and leisure plays a crucial role in whether retirement flexibility increases or decreases the demand for stocks. From the lowest panel of Figure 7, it can be seen that this parameter is also decisive in the direction of the general-equilibrium effects. This panel is based on an elasticity of substitution of one half, implying that income effects now dominate substitution effects (i.e., $\pi_{z,A} < 0$). That means, a negative productivity shock induces people to retire later in time. In general equilibrium, this retirement shift moderates the direct drop in the capital return due to the negative productivity shock. In other words, when income effects are dominating, the sensitivity of the capital return to productivity risk ($\pi_{r_k,A}$) is lower in general equilibrium than in partial equilibrium. Because this lower sensitivity increases the insurance effect of retirement flexibility, the relative equity share is now increasing in the degree of productivity risk.

To summarize, the equity exposure can either be higher or lower in general equilibrium than in partial equilibrium. This is true both for productivity and depreciation risk. With depreciation risk, the labour supply elasticity with respect to shocks is lower in general equilibrium (which depresses equity investments) but, at the same time, the capital return is less sensitive to these shocks (which stimulates equity investments). We have shown that for low (high) levels of risk aversion the first (second) effect is dominating. In case of productivity risk, the elasticity of intratemporal substitution determines whether agents invest more or less in equity in general equilibrium compared to partial equilibrium. For high intratemporal substitution (substitution effect dominates), the capital return is relatively more sensitive to productivity shocks in general equilibrium resulting in lower equity exposures. For low substitution (income effect dominates), the opposite holds, meaning that agents invest relatively more in equity in general equilibrium.

6 Conclusion

In this paper, we have developed a stochastic general-equilibrium model with two overlapping generations. The model is used to analyse the interaction between consumption, portfolio choice and retirement decisions. In the literature, retirement flexibility is often viewed as a kind of insurance against bad investment outcomes. This paper reviews this

benchmark result in a more general model. In particular, in our model the risk factors (productivity risk and depreciation risk) are directly linked to the production structure of the economy. Second, and more importantly, we combine a partial-equilibrium approach with a general-equilibrium approach thereby explicitly recognizing that correlations between productivity and depreciation shocks are endogenous. Finally, we allow for more general preferences which are characterized by a constant elasticity of substitution (CES) function of consumption and leisure.

Our main findings are as follows. First, the relevance of retirement flexibility as a hedging instrument strongly depends on the type of risk agents are subject to. Productivity risk affects wages and asset returns in the same direction. Under retirement flexibility, this positive correlation between wages and asset returns is reinforced by the substitution effect on labour supply resulting in a *lower* preference for risk taking. In partial equilibrium this lower demand leads to lower equity shares in the total investment portfolio while in general equilibrium it leads to higher equity premia as the supply of assets is (partly) fixed. With depreciation risk, though, wages are only indirectly affected by general-equilibrium effects. In this case, the income effect dominates implying that labour income and capital returns are negatively correlated which leads to a *higher* preference for risk taking. In partial equilibrium, this higher demand leads to higher portfolio shares invested in equity, in general equilibrium it leads to lower equity premia.

Second, our analysis reveals that the elasticity of substitution between consumption and leisure is of crucial importance in determining to which extent retirement flexibility protects retirees against bad investment returns. Indeed, this elasticity governs the relative strength of income and substitution effects in labour supply and therefore determines the hedging effect of retirement flexibility. Our analysis clearly shows that the advantage of flexible retirement as a hedging instrument is smaller if substitution effects are relatively important. Empirical studies indeed suggest that substitution effects are more important for the retirement decision than income or wealth effects.

Finally, we find that general-equilibrium effects play an important role in the interaction between portfolio choice and retirement. Ignoring these effects by sticking to a partial-equilibrium framework can either overstate or understate the insurance benefits of retirement flexibility. It is mainly the degree of substitution between consumption and leisure that determines the direction of the general-equilibrium effects. For high substitution elasticities, which seems empirically the most relevant case, labour supply behaviour amplifies the sensitivity of capital returns to productivity risk making retirement flexibility less effective as hedging tool in general equilibrium than in partial equilibrium.

Our paper can benefit from a number of relevant extensions. First, the menu of shocks could be extended to include, for example, demographic shocks (such as shocks to longevity or fertility) and inflation shocks (see e.g., Adema, 2008). As a second extension, we can include social security along with individual heterogeneity. Retirement flexibility and social security have in common that they both can protect retirees against adverse shocks. In this paper, we have deliberately focused on a simple setting without social security thereby ignoring the interaction between retirement flexibility and social security. In future work, we want to introduce social security along with individual heterogeneity to tackle similar issues as studied in this paper. We will in particular focus on how portfolio and retirement decisions, made by heterogeneous agents, are affected by uniform social security systems.

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A The steady state

A.1 Derivation first-order conditions

We can write equation (T1.3) as,

$$\begin{aligned} 1 &= \mathrm{E}_t\left[\exp\left\{\log\beta + \log\left(1+r_{k,t+1}\right) + \theta\log c_{1,t} - \rho\log c_{2,t+1} + (\rho-\phi)\log v_{t+1}\right\}\right] \\ &\equiv \mathrm{E}_t\left[\exp\left\{x_{t+1}\right\}\right] \end{aligned} \tag{A.1}$$

Taking a second-order Taylor expansion of $\exp\{x_{t+1}\}$ around $\mathrm{E}_t\, x_{t+1} \equiv \bar{x}_t$, we obtain,

$$\begin{aligned} 1 &\approx \mathrm{E}_t\left[\exp\left\{\bar{x}_t\right\}\left(1 + x_{t+1} - \bar{x}_t + \frac{1}{2}\left(x_{t+1}-\bar{x}_t\right)^2\right)\right] \\ &= \exp\left\{\bar{x}_t\right\}\left(1 + \frac{1}{2}\mathrm{Var}_t x_{t+1}\right) \end{aligned} \tag{A.2}$$

Then, a first-order Taylor expansion around zero gives the result,

$$\begin{aligned} 1 &\approx 1 + \bar{x}_t + \frac{1}{2}\mathrm{Var}_t x_{t+1} \quad \Rightarrow \\ 1 &\approx \exp\left\{\bar{x}_t + \frac{1}{2}\mathrm{Var}_t x_{t+1}\right\} \end{aligned} \tag{A.3}$$

Note that we can write equation (7) as,

$$\log v = \frac{\log\left[\exp\left\{\log(1-\gamma) + (1-\rho)\log c_1\right\} + \exp\left\{\log\gamma + (1-\rho)\log(1-z)\right\}\right]}{(1-\rho)(1-\gamma)} \tag{A.4}$$

Taking a first-order Taylor expansion around zero then gives:

$$\log v \approx \log c_1 + \eta\log(1-z) \tag{A.5}$$

with $\eta \equiv \gamma/(1-\gamma)$. Combining equations (A.3) and (A.5), we obtain the steady-state Euler equation regarding capital investments, equation (T2.3):

$$c_1^{-\theta} = \beta\left(1+r_k\right)c_2^{-\phi}(1-z)^{\eta(\rho-\phi)}\exp\left(\frac{1}{2}\sigma^2_{r_k-v}\right) \tag{A.6}$$

with $\sigma^2_{r_k-v}$ defined in equation (18).

The derivation of the second Euler equation, equation (T2.4), and of the optimality condition with respect to fixed retirement, equation (T2.7b), are similar to the one above.

A.2 Deterministic steady state

Suppose that $\theta = \rho \to 1$ and $\delta = 1$. Ignoring the risk terms or assuming a non-stochastic steady state implies that $r_k = r_b \equiv r$. Then inserting equation (T2.1) and equation (T2.2) in the Euler equation (T2.3) (or equation (T2.4)) gives:

$$\frac{1+\beta}{\beta}k = w - rb - \frac{1+\beta}{\beta}b - \frac{w}{(1+r)\beta}z \tag{A.7}$$

From the optimality condition with respect to leisure, equation (T2.7a) (or equation (T2.7b)), we derive:

$$k = \frac{w}{(1+r)\eta}(1-z) - \frac{w}{1+r}z - b \tag{A.8}$$

Substituting equation (A.8) in (A.7) and solving for z gives:

$$z = \frac{1+\beta-\beta\eta(1+r)\left(1-\frac{rb}{w}\right)}{1+\beta+\beta\eta} \tag{A.9}$$

Inserting equation (A.8) in equation (A.7) and solving for k leads to:

$$k = \frac{\beta(1+\eta)w\left(1-\frac{rb}{w}\right) - \frac{w}{1+r} - (1+\beta+\beta\eta)b}{1+\beta+\beta\eta} \tag{A.10}$$

Using the factor prices, equation (T2.5) and equation (T2.6), we can rewrite equation (A.10) into:

$$1+z = \frac{\beta(1+\eta)\left(1-\frac{rb}{w}\right)(1-\alpha)\left(\frac{k}{1+z}\right)^{\alpha-1} - \frac{1-\alpha}{\alpha}}{(1+\beta+\beta\eta)\left(1+\frac{b}{k}\right)} \tag{A.11}$$

In the same way, we can rewrite (A.9) into:

$$1+z = \frac{2(1+\beta)+\beta\eta-\beta\eta\left(1-\frac{rb}{w}\right)\alpha A\left(\frac{k}{1+z}\right)^{\alpha-1}}{1+\beta+\beta\eta} \tag{A.12}$$

Equations (A.11) and (A.12) form a closed system in k and z. Solving these equations gives for the capital-labour ratio,

$$\frac{k}{1+z} = \left[\frac{\left(1-\alpha+\eta+\eta\alpha\frac{b}{k}\right)\alpha\beta\left(1-\frac{rb}{w}\right)}{1-\alpha+\left(1+\frac{b}{k}\right)\alpha(2+2\beta+\beta\eta)}\right]^{\frac{1}{1-\alpha}} \tag{A.13}$$

and for labour supply:

$$z = \frac{1-\alpha-\alpha\eta-\alpha\eta\frac{b}{k}}{1+\eta-\alpha+\alpha\eta\frac{b}{k}} \tag{A.14}$$

Using the definition $\lambda \equiv k/(b+k)$ in equation (A.14), gives the labour supply decision as function of the portfolio choice (equation (22)). Notice that equation (A.13) still depends on w and r, which are functions of the capital-labour ratio. Again using equations (T2.5) and (T2.6), we derive:

$$\frac{rb}{w} = \frac{\alpha A \left(\frac{k}{1+z}\right)^{\alpha-1} - 1}{(1-\alpha)\left(\frac{k}{1+z}\right)^{\alpha-1}} \frac{b}{k}(1+z) \tag{A.15}$$

Finally, substituting this expression in equation (A.13) and using equation (A.14), we obtain:

$$\frac{k}{1+z} = \left[\frac{\alpha\beta A(1+\eta-\alpha-2\alpha\frac{b}{k})}{1+\alpha+\alpha\beta(2+\eta)+2\alpha\frac{b}{k}}\right]^{\frac{1}{1-\alpha}} \tag{A.16}$$

Using the definition λ in equation (A.16), gives the capital-labour ratio as function of the portfolio choice (equation (23)).

B The partial elasticities

We are looking for the following dynamic system:

$$\hat{k}_{t+1} = \pi_{k,k}\hat{k}_t + \pi_{k,A}\omega_{A,t} + \pi_{k,\delta}\omega_{\delta,t} \tag{A.17}$$

and:

$$\begin{bmatrix} \hat{c}_{1,t} \\ \hat{c}_{2,t} \\ \hat{r}_{k,t} \\ \hat{w}_t \\ \hat{r}_{b,t+1} \\ \hat{z}_t \text{ or } \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \pi_{c_1,k} \\ \pi_{c_2,k} \\ \pi_{r_k,k} \\ \pi_{w,k} \\ \pi_{r_b,k} \\ \pi_{z,k} \end{bmatrix} \hat{k}_t + \begin{bmatrix} \pi_{c_1,A} & \pi_{c_1,\delta} \\ \pi_{c_2,A} & \pi_{c_2,\delta} \\ \pi_{r_k,A} & \pi_{r_k,\delta} \\ \pi_{w,A} & \pi_{w,\delta} \\ \pi_{r_b,A} & \pi_{r_b,\delta} \\ \pi_{z,A} & \pi_{z,\delta} \end{bmatrix} \begin{bmatrix} \omega_{A,t} \\ \omega_{\delta,t} \end{bmatrix} \tag{A.18}$$

where $\pi_{x,y}$ denotes the partial elasticity of endogenous variable x with respect to state variable y. With retirement flexibility, the recursive law for labour supply is based on $\hat{z}_t$. With retirement inflexibility, it is based on $\hat{z}_{t+1}$ because retirement is predetermined at time t.

B.1 Flexible retirement

Note that equations (T3.2), (T3.5), (T3.6) and (T3.7a) form an independent system of the endogenous variables $\hat{c}_{2,t}$, $\hat{w}_t$, $\hat{r}_{k,t}$ and $\hat{z}_t$ in the predetermined variables $\hat{k}_t$ and $\hat{r_{b,t}}$ and

the exogenous shocks $\omega_{A,t}$ and $\omega_{\delta,t}$. From this system we can infer the partial elasticities with respect to productivity shocks and depreciation shocks:

$$\begin{aligned}
\pi_{c_2,A} &= \frac{(1-z+\rho z+\alpha\rho)\,y}{c_2\Delta} > 0 \\
\pi_{r_k,A} &= \frac{(r_k+\delta)\,(\rho+\rho z+\rho\Gamma+1-z)}{r_k\Delta} > 0 \\
\pi_{w,A} &= \frac{\rho\,(1+z)\,(1+\Gamma-\alpha)}{(1-\alpha)\Delta} > 0 \\
\pi_{z,A} &= \frac{(1+z)\,[(1-z)(1-\alpha)-\rho\Gamma\,(\alpha+z)]}{z(1-\alpha)\Delta} \\
\pi_{c_2,\delta} &= -\frac{\delta k\,(\rho+\alpha-\alpha z+\rho z)}{c_2\Delta} < 0 \\
\pi_{r_k,\delta} &= -\frac{\delta\,[\rho(1+z)+(1-z)\alpha+\rho\Gamma\,(1+z-\alpha z)]}{r_k\Delta} < 0 \\
\pi_{w,\delta} &= -\frac{\rho(1-z)\delta k\alpha}{c_2\Delta} < 0 \\
\pi_{z,\delta} &= \frac{(1+z)(1-z)\rho\delta k}{c_2 z\Delta} > 0
\end{aligned}$$

To save on notation, Γ and Δ are defined as:

$$\begin{aligned}
\Gamma &\equiv w^{1-\frac{1}{\rho}}\eta^{\frac{1}{\rho}} \\
\Delta &\equiv (1-z)\alpha+(1+z)\rho(1+\Gamma)+\rho\alpha\Gamma
\end{aligned}$$

Note that the sign of $\pi_{z,A}$ is ambiguous; it can either be positive or negative, depending on the substitution between consumption and leisure.

Noting that $\mathrm{E}_t\,\omega_{A,t+1} = \mathrm{E}_t\,\omega_{\delta,t+1} = 0$ and using the Euler equations (T3.3) and (T3.4), we now can express the bond return $\hat{r}_{b,t+1}$, the conditional expectations $\mathrm{E}_t\,\hat{c}_{2,t+1}$ and $\mathrm{E}_t\,\hat{c}_{r_k,t+1}$ together with first-period consumption $\hat{c}_{1,t}$ as functions of the next-period capital stock $\hat{k}_{t+1}$:

$$\Phi_{r_b} \equiv \frac{\hat{r}_{b,t+1}}{\hat{k}_{t+1}} = -\frac{(1+r_b)\,\rho(1+z)y\,[(r_k+\delta)\,(1+\Gamma-\alpha)+(1-\delta)\alpha\Gamma]}{(1+r_k)\,r_b y\Delta+(1+r_b)\,r_b\rho\,(r_k+\delta)\,\Gamma\,(1+z)\,b} \tag{A.19}$$

$$\begin{aligned}
\Phi_{c_2} \equiv \frac{\mathrm{E}_t\,\hat{c}_{2,t+1}}{\hat{k}_{t+1}} &= \frac{[\rho+\alpha+z(\rho-\alpha)]\,[(1-\delta)k+r_b b\Phi_{r_b}]}{c_2\Delta} \\
&\quad+\frac{\alpha\,[1-z+\rho(z+\alpha)]\,y}{c_2\Delta}
\end{aligned} \tag{A.20}$$

$$\Phi_z \equiv \frac{\mathrm{E}_t\,\hat{z}_{t+1}}{\hat{k}_{t+1}} = \frac{(1-z)(1+z)\left[\alpha c_2 - \alpha\rho(y-w) - \rho(1-\delta)k - \rho r_b b \Phi_{r_b}\right]}{c_2 z \Delta} \quad \text{(A.21)}$$

$$\Phi_{c_1} \equiv \frac{\hat{c}_{1,t}}{\hat{k}_{t+1}} = \frac{1}{\theta}\left[\phi\Phi_{c_2} - \frac{r_b\Phi_{r_b}}{1+r_b} + \frac{\eta(\rho-\phi)z\Phi_z}{1-z}\right] \quad \text{(A.22)}$$

Notice from equation (A.19) that $\hat{r}_{b,t}$ and $\hat{k}_t$ - the two predetermined variables - move proportionally. Therefore, using this equation, we can substitute out $\hat{r}_{b,t}$ from the state space.

Substituting equation (A.22) in the budget restriction, equation (T3.1), we ultimately obtain the solution to equation (A.17), with:

$$\begin{aligned}
\pi_{k,k} &= \frac{w\pi_{w,k} - r_b b \Phi_{r_b}}{c_1\Phi_{c_1} + k} \\
\pi_{k,A} &= \frac{w\pi_{w,A}}{c_1\Phi_{c_1} + k} \\
\pi_{k,\delta} &= \frac{w\pi_{w,\delta}}{c_1\Phi_{c_1} + k}
\end{aligned}$$

The system is stable if and only if $\pi_{k,k} < 1$. This solution for the endogenous state variable pins down the solutions of the other endogenous variables in equation (A.18). The partial elasticities with respect to the capital stock are equal to:

$$\begin{aligned}
\pi_{c_1,k} &= \Phi_{c_1}\pi_{k,k} \\
\pi_{c_2,k} &= \Phi_{c_2} \\
\pi_{r_k,k} &= \frac{r_k+\delta}{r_k}\left[\frac{\alpha(\rho+\rho z+\rho\Gamma+1-z)}{\Delta} - \frac{\Gamma\rho(1+z)(k-\delta k+r_b b\Phi_{r_b})}{y\Delta} - 1\right] \\
\pi_{w,k} &= \frac{\alpha\rho(1+z)(1+\Gamma-\alpha)}{(1-\alpha)\Delta} + \frac{\alpha\rho(1-z)(k-\delta k+r_b b\Phi_{r_b})}{c_2\Delta} \\
\pi_{r_b,k} &= \Phi_{r_b}\pi_{k,k} \\
\pi_{z,k} &= \Phi_z
\end{aligned}$$

The remaining elasticities with respect to productivity and depreciation shocks are:

$$\begin{aligned}
\pi_{c_1,A} &= \Phi_{c_1}\pi_{k,A} \\
\pi_{r_b,A} &= \Phi_{r_b}\pi_{k,A} \\
\pi_{c_1,\delta} &= \Phi_{c_1}\pi_{k,\delta} \\
\pi_{r_b,\delta} &= \Phi_{r_b}\pi_{k,\delta}
\end{aligned}$$

B.2 Fixed retirement

In case of fixed retirement, equations (T3.2), (T3.5) and (T3.6) form an independent system of the endogenous variables $\hat{c}_{2,t}$, $\hat{w}_t$ and $\hat{r}_{k,t}$ in terms of the three predetermined variables $\hat{k}_t$, $\hat{r_{b,t}}$ and $\hat{z}_t$ and the two exogenous shocks $\omega_{A,t}$ and $\omega_{\delta,t}$. From this system we can derive the following elasticities with respect to productivity:

$$\begin{aligned}
\pi_{c_2,A} &= \frac{y-w}{c_2} > 0 \\
\pi_{r_k,A} &= \frac{r_k+\delta}{r_k} > 0 \\
\pi_{w,A} &= 1 \\
\pi_{c_2,\delta} &= -\frac{\delta k}{c_2} < 0 \\
\pi_{r_k,\delta} &= -\frac{\delta}{r_k} < 0 \\
\pi_{w,\delta} &= 0
\end{aligned}$$

with Γ now defined as:

$$\Gamma \equiv w^{1-\frac{1}{\rho}} \eta^{\frac{1}{\rho}} \exp\left[\frac{1}{2\rho}\left(\sigma_{c_2}^2 - \sigma_{w-c_2}^2\right)\right]$$

With inflexible retirement, equation (A.19)-(A.22) are still valid but Φ_z is now defined as $\Phi_z \equiv \hat{z}_t/\hat{k}_t$. Consequently, also the partial elasticities with respect to the capital stock still hold except that $\pi_{z,k} = \Phi_z \pi_{k,k}$. The remaining elasticities with respect to productivity shocks are:

$$\begin{aligned}
\pi_{k,A} &= \frac{w}{c_1\Phi_{c_1} + k} \\
\pi_{c_1,A} &= \Phi_{c_1}\pi_{k,A} \\
\pi_{r_b,A} &= \Phi_{r_b}\pi_{k,A} \\
\pi_{z,A} &= \Phi_z \pi_{k,A}
\end{aligned}$$

With fixed retirement, the capital stock, first-period consumption, the bond return and labour supply do not respond to depreciation shocks. That is,

$$\pi_{k,\delta} = \pi_{c_1,\delta} = \pi_{r_b,\delta} = \pi_{z,\delta} = 0$$

C Retirement as hedge

Suppose that we have log-linear life-time utility in consumption and leisure (i.e., $\rho = \theta = 1$). Assume further that wages are non-stochastic.

C.1 Flexible retirement

Portfolio choice. Inserting equation (12) in equation (10), and using equation (8), we obtain:

$$c_{2,t+1} = \frac{1}{1+\eta}\left(1 + r_{T,t+1}\right)\left(s_t + \frac{w_{t+1}}{1 + r_{b,t+1}}\right) \tag{A.23}$$

where $r_{T,t+1}$ is defined in equation (32). Note that $c_{2,t+1}$ is decomposed in non-stochastic terms (the first and third term) and a stochastic term (the second one). Substituting (A.23) in the two Euler equations (for $j = r_b$ and $j = r_k$) and subtracting both, we have:

$$\mathrm{E}_t\left[\left(1 + r_{T,t+1}\right)^{-1}\left(r_{k,t+1} - r_{b,t+1}\right)\right] = 0 \tag{A.24}$$

Taking logs of equation (A.24), we obtain:

$$\mathrm{E}_t\,\tilde{r}_{k,t+1} + \frac{1}{2}\mathrm{Var}_t\tilde{r}_{k,t+1} - \tilde{r}_{b,t+1} = \mathrm{Cov}_t\left(\tilde{r}_{T,t+1}, \tilde{r}_{k,t+1}\right) \tag{A.25}$$

with $\tilde{r}_i \equiv \log\left(1 + r_i\right)$ and $i = k, T$ and where we used the Jensen's inequality condition for a lognormal variable, i.e., $\log \mathrm{E}_t\, x_{t+1} = \mathrm{E}_t \log x_{t+1} + 1/2\mathrm{Var}_t \log x_{t+1}$. To derive the term on the left-hand side of equation (A.25), we follow Campbell and Viceira (2002) and use a second-order Taylor approximation of the portfolio return, equation (32). This gives,

$$\tilde{r}_{T,t+1} \approx \tilde{r}_{b,t+1} + a_t\left(\tilde{r}_{k,t+1} - \tilde{r}_{b,t+1}\right) + \frac{1}{2}a_t(1 - a_t)\mathrm{Var}_t\tilde{r}_{k,t+1} \tag{A.26}$$

Hence,

$$\mathrm{Cov}_t\left(\tilde{r}_{T,t+1}, \tilde{r}_{k,t+1}\right) = a_t\mathrm{Var}_t\tilde{r}_{k,t+1} \tag{A.27}$$

Substituting equation (A.27) into (A.25) then gives:

$$a_t = \frac{\mathrm{E}_t\,\tilde{r}_{k,t+1} - \tilde{r}_{b,t+1} + \frac{1}{2}\mathrm{Var}_t\tilde{r}_{k,t+1}}{\mathrm{Var}_t\tilde{r}_{k,t+1}} \tag{A.28}$$

Finally, inserting (A.28) in (33), we end up with the portfolio allocation in terms of financial wealth (see equation (29)).

Consumption and leisure. Substituting equation (A.23) in equation (11) (for $j = r_b$) and rearranging gives:

$$c_{1,t}^{-1} = \beta(1+\eta)\left(1+r_{b,t+1}\right)\mathrm{E}_t\left(1+r_{T,t+1}\right)^{-1}\left(w_t - \tau_t - c_{1,t} + \frac{w_{t+1}}{1+r_{b,t+1}}\right)^{-1} \tag{A.29}$$

Notice that:

$$\begin{aligned}\left(1+r_{b,t+1}\right)\mathrm{E}_t\left(1+r_{T,t+1}\right)^{-1} &= \left(1+r_{b,t+1}\right)\mathrm{E}_t\left(1+r_{T,t+1}\right)^{-1} \\ &\quad + a_t\,\mathrm{E}_t\left[\left(1+r_{T,t+1}\right)^{-1}\left(r_{k,t+1}-r_{b,t+1}\right)\right] \\ &= 1 \end{aligned} \tag{A.30}$$

Hence, first-period consumption satisfies:

$$c_{1,t} = \frac{1}{1+\beta(1+\eta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1+r_{b,t+1}}\right) \tag{A.31}$$

Note that the propensity to consume is the same as under certainty. Hence, there is no precautionary saving motive, which is a direct implication of the log-utility specification (see Sandmo, 1970). Combining (A.31) and (A.23), we obtain for second-period consumption:

$$c_{2,t+1} = \frac{\beta\left(1+r_{T,t+1}\right)}{1+\beta(1+\eta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1+r_{b,t+1}}\right) \tag{A.32}$$

Substituting (A.32) in (12), we obtain the expression for labour supply (see equation (31)).

C.2 Inflexible retirement

Portfolio choice. Consider now the fixed retirement setting. Then the intertemporal budget constraint becomes:

$$c_{2,t+1} = \left(1+r_{T,t+1}\right)\left(s_t + \frac{w_{t+1}z_{t+1}}{1+r_{b,t+1}}\right) \tag{A.33}$$

with $r_{T,t+1}$ again defined as in (32) but where a_t now satisfies:

$$a_t = \frac{\lambda_t s_t}{s_t + \frac{w_{t+1}z_{t+1}}{1+r_{b,t+1}}} \tag{A.34}$$

Inserting (A.33) in the two Euler equations (for $j = r_b$ and $j = r_k$) again gives condition (A.24). Hence, a_t is still given by equation (A.28). Inserting (A.28) into (33) we end up

with the portfolio share in terms of financial wealth (see equation (30)).

Consumption and leisure. The fact that wages are nonstochastic implies that the first-order condition with respect to leisure consumption, equation (13), becomes:

$$\frac{\eta}{1-z_{t+1}} = w_{t+1}\,\mathrm{E}_t\,c_{2,t+1}^{-1} \tag{A.35}$$

Combining (A.35) and (11) (for $j = r_b$), gives:

$$(1-z_{t+1})w_{t+1} = \eta\beta\,(1+r_{b,t+1})\,c_{1,t} \tag{A.36}$$

Substituting (A.33) in (11) (again for $j = r_b$) and rearranging gives:

$$c_{1,t}^{-1} = \beta\left(w_t - \tau_t - c_{1,t} + \frac{w_{t+1}z_{t+1}}{1+r_{b,t+1}}\right)^{-1} \tag{A.37}$$

where we (again) used equality (A.30). Substitution of (A.36) in (A.37) gives:

$$c_{1,t}^{-1} = \beta\left[w_t - \tau_t + \frac{w_{t+1}}{1+r_{b,t+1}} - (1+\eta\beta)c_{1,t}\right]^{-1} \tag{A.38}$$

Hence,

$$c_{1,t} = \frac{1}{1+\beta(1+\eta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1+\tau_{b,t+1}}\right) \tag{A.39}$$

Note that consumption (and thus savings) under fixed labour supply is exactly equal to consumption under flexible labour supply. Substituting (A.39) in (A.36) and solving for z_{t+1}, we ultimately obtain the optimal retirement decision (see equation (34)).

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