CPB Memorandum

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The derivatives of complex characteristic roots in the econometric modelling textbook of Kuh et al.

The computation of the derivatives of the modulus and cycle time of complex characteristic roots (eigenvalues) of dynamic economic models is not discussed adequately in the textbook of Kuh et al., 1985, "Structural Sensitivity in Econometric Models".

1 Introduction

Characteristic roots, or eigenvalues, are a useful tool in the study of linear dynamic economic models. The relation between the roots and the model coefficients are expressed in the derivatives of the former with respect to the latter.

In general, the roots are complex numbers. This note discusses the derivatives of both the modulus and the cycle time of the complex roots.

An important reference on this subject is Kuh et al. (1985) [hereafter: KNH]. Below it is shown that KNH discusses these derivatives unsatisfactory.

In the next two sections the model and the derivatives of the roots are given. Section 4 presents the formulas for the derivatives of the modulus and the cycle time of complex roots. The substance of the paper is section 5, with a critical comment on the discussion of these derivatives in KNH. The last section gives a short conclusion.

2 The model

Let the linear model be as follows, with the exogenous variables omitted:

$$Dy_t = Ey_{t-1} \tag{2.1}$$

This can be written as

$$y_t = Ay_{t-1} \tag{2.2}$$

with

$$A \equiv D^{-1}E \tag{2.3}$$

The characteristic roots of the system are the eigenvalues λ_h $(h = 1, \ldots, n)$ of the matrix A, where n is the order of A. We assume that there are no multiple eigenvalues.

3 Derivatives of eigenvalues

Without loss of generality we choose the scaling of the eigenvectors such that

$$\ell_h{}' r_h = 1 \tag{3.1}$$

where ℓ_h and r_h are the left eigenvector and the right eigenvector of λ_h, respectively[1]. Then the derivatives of λ_h with respect to the coefficients of the model matrices are as follows:

$$\frac{\partial \lambda_h}{\partial a_{ij}} = (\ell_h)_i (r_h)_j \tag{3.2}$$

$$\frac{\partial \lambda_h}{\partial e_{ij}} = \left(\ell_h{}' D^{-1}\right)_i (r_h)_j \tag{3.3}$$

$$\frac{\partial \lambda_h}{\partial d_{ij}} = -\lambda_h \frac{\partial \lambda_h}{\partial e_{ij}} \tag{3.4}$$

See KNH: the equations (2.3.15) or (2B.8.10), (2B.8.17), and (2B.8.16), respectively. In general, these derivatives are complex values.

Formula (3.2) can also be found in other publications of similar age. See for instance Magnus and Neudecker (1988), chapter 8 (section 7 and further) and the bibliographical notes on page 169. In the engineering literature, see for instance the references in Zhang and Wen (2005)[2].

4 Derivatives of modulus and cycle time

In practice one is interested in the derivatives of the coordinates of the complex eigenvalues; either the cartesian coordinates (the real part and the imaginary part) or the polar coordinates (the modulus and the angle with the real axis). Instead of the angle, we choose the cycle time, which is easier to interpret. These derivatives are given below.

These formulas are true for any nonzero complex number λ_h and any real number x. The value of any one of the derivatives (3.2), (3.3) and (3.4) above can be substituted for $\partial \lambda_h / \partial x$ in the right-hand side of these formulas.

$$\frac{\partial \mathrm{Re}\lambda_h}{\partial x} = \mathrm{Re}\frac{\partial \lambda_h}{\partial x} \tag{4.1}$$

$$\frac{\partial \mathrm{Im}\lambda_h}{\partial x} = \mathrm{Im}\frac{\partial \lambda_h}{\partial x} \tag{4.2}$$

$$\frac{\partial |\lambda_h|}{\partial x} = \frac{1}{|\lambda_h|} \left(\mathrm{Re}\lambda_h \ \mathrm{Re}\frac{\partial \lambda_h}{\partial x} + \mathrm{Im}\lambda_h \ \mathrm{Im}\frac{\partial \lambda_h}{\partial x} \right) \tag{4.3}$$

$$\frac{\partial \tau(\lambda_h)}{\partial x} = \frac{\tau^2}{2\pi |\lambda_h|^2} \left(\mathrm{Im}\lambda_h \ \mathrm{Re}\frac{\partial \lambda_h}{\partial x} - \mathrm{Re}\lambda_h \ \mathrm{Im}\frac{\partial \lambda_h}{\partial x} \right) \tag{4.4}$$

with

$$|\lambda_h| = \sqrt{\mathrm{Re}^2 \lambda_h + \mathrm{Im}^2 \lambda_h} \tag{4.5}$$

[1] The inner product in equation (3.1) is simply $\sum (\ell_h)_i (r_h)_i$, and not the usual complex inner product where $u'v$ is the complex conjugate of $v'u$ and where $v'v$ is positive for any nonzero complex vector v.

[2] Equation (3) on page 162 of Magnus and Neudecker (1988) is a generalization of our equation (3.2) to the case of a complex matrix; note that our left eigenvector ℓ_h is the complex conjugate of their vector v_0, which is defined as in the LAPACK system. Equation (18) of Zhang and Wen (2005) is also a generalization of our equation (3.2); note that in our case $M = \mathrm{I}$ and hence $\partial M / \partial V = 0$.

and

$$\tau(\lambda_h) = \frac{2\pi}{\text{the angle with the real axis}} = \frac{2\pi}{\arctan\left(\mathrm{Im}\lambda_h/\mathrm{Re}\lambda_h\right)} \tag{4.6}$$

5 The book

The treatment of the derivatives of the modulus and the cycle time in KNH is somewhat odd.

First, the *modulus* is discussed on page 68. Below equation (2B.8.17) it is suggested that this equation, and the related equations which are also shown in our section 3 above, are “modulus root sensitivities”, which they are not. They are the complex root sensitivities themselves. The derivative of the modulus occurs in the elasticities (2B.8.18) and further, without any indication how to compute this.

Our modulus equation (4.3) is missing in KNH. Note that this is not simply $\partial|\lambda_h|/\partial x = |\partial\lambda_h/\partial x|$, similar to the trivial equations (4.1) and (4.2) above.

Second, the *cycle time* τ is discussed on page 69. Our equation (4.4) is the same as equation (2B.8.26) in KNH.

The next equation, (2B.8.27), is the result of the substitution of derivative (3.2) above into the right-hand side of equation (4.4). This is presented as an example of how to compute this for all three model matrices. The result is expressed in terms of the real and imaginary parts of the eigenvectors. However, there is nothing in the nature of the cycle time that requires the use of the real and imaginary parts of the eigenvectors once the equations (3.2), (3.3) and (3.4) are computed.

Finally, we note that equation (2B.8.27) contains an error: the plus sign between parentheses in the last line must be a minus sign[3].

6 Conclusion

The symmetry of the derivatives of the modulus and the cycle time of complex roots (eigenvalues), as shown in our equations (4.3) and (4.4), is overlooked in KNH. The formula for the derivative of the modulus is missing and in the discussion of the derivative of the cycle time there is an equation which is useless and erroneous.

[3] This was pointed out by Evelyn van Lochem when she was at the CPB.

References

Kuh, E., J.W. Neese and P. Hollinger, 1985, *Structural Sensitivity in Econometric Models*, Wiley, New York.

Magnus, J.R. and H. Neudecker, 1988, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Wiley, New York.

Zhang, Y. and B. Wen, 2005, Multi-dimensional sensitivity analysis of eigen-systems, *International Journal of Applied Mathematics and Mechanics*, vol. 1, pp. 97–105, http://ijamm.bc.cityu.edu.hk.