The relation between the differential land rent, commuting cost, and differential wage sum in a monocentric city

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1 Case 1: all workers work in the CBD

Consider a simple world, where all work in a city is concentrated in the CBD which is surrounded by suburbs, where all people live who work in the CBD. Nobody works in the countryside. The land intensity of work is zero and the land intensity of living is constant: $1/\alpha$. Hence, the surface of the CBD is zero. Let ω be the wage surplus of the CBD above the countryside and let γ be the commuting cost per unit of distance. Define $R(s)$ to be the differential land rent (i.e.: land rent above land rent in the countryside) at distance s from the CBD and let S be the distance from the city edge to the CBD.The land rent equalizes the difference in commuting cost to the CBD. At the city edge, workers are indifferent between working in the CBD and working in the countryside. Hence:

$$\begin{aligned} R(s) &= \alpha\gamma (S - s), \\ \gamma S &= \omega. \end{aligned}$$

Hence, the total workforce in the CBD L, the total commuting cost TC, the total diffential land rent TDR, and the total differential wage sum TDW satisfy:

$$\begin{aligned} L &= 2\pi \int_0^S \alpha s ds = \pi\alpha S^2, \\ TC &= 2\pi \int_0^S \alpha\gamma s^2 ds = \frac{2}{3}\pi\alpha\gamma S^3, \\ TDR &= 2\pi \int_0^S \alpha\gamma s (S - s) ds = \frac{1}{3}\pi\alpha\gamma S^3, \\ TDW &= \omega L = \pi\alpha\gamma S^3. \end{aligned}$$

Hence, the following proposition holds:

Proposition 1 *The relation between total commuting cost, the total differential land rent, and the total differential wage sum is:*

$$TDR = \frac{1}{3}TDW = \frac{1}{2}TC.$$

Claim 2 *These expression are first order Taylor approximations of the relations that hold in more general models, where commuting cost vary non-linearly with distance and where the land intensity is negatively related to the land rent and hence positively related to the distance to the CBD.*

2 Case 2: part of the workers work in the suburbs

We add a slight complication to this world. Suppose that part of the workforce is employed in the suburbs. Like in the CBD, working in the suburbs has a zero land intensity. One can think of these workers to provide non-tradable services that are tied to the location of living, like retail trade. Let λ denote the fraction of workforce that works in the suburbs at the location where they live, while the remaining fraction works in the CBD. Define $W(s)$ the wage surplus above the wage at the countryside that employers pay at location s. The wage surplus in the CBD, $W(0)$, is by definition equal to ω. Since somebody living at location s must be indifferent between working in the CBD and working at location s and since the commuting cost to the CBD are equal to γs, $W(s)$ reads:

$$W(s) = \gamma(S - s),$$

where again $\gamma S = \omega$. The workers living at location s buy non-tradables from the share λ of the workers who work in the non-tradable sector at a price $\lambda W(s) = \lambda\gamma(S - s)$ and the pay a rent $\alpha^{-1}R(s)$. Workers working in the CBD must be indifferent between living at various locations. Hence, the wage surplus γS in the CBD must be equal to the rent, the cost of living and the commuting cost at each location s:

$$\begin{aligned} \gamma S &= \alpha^{-1}R(s) + \lambda\gamma(S - s) + \gamma s \Rightarrow \\ R(s) &= (1 - \lambda)\alpha\gamma(S - s). \end{aligned}$$

The previous relations for L, TC, TDR, and TDW have to be modified as follows:

$$
\begin{aligned}
L &= 2\pi \int_0^S (1-\lambda)\,\alpha s ds = \pi (1-\lambda)\,\alpha S^2, \\
TC &= 2\pi \int_0^S (1-\lambda)\,\alpha \gamma s^2 ds = \frac{2}{3}\pi (1-\lambda)\,\alpha\gamma S^3, \\
TDR &= 2\pi \int_0^S (1-\lambda)\,\alpha\gamma s\,(S-s)\,ds = \frac{1}{3}\pi (1-\lambda)\,\alpha\gamma S^3, \\
TDW &= \omega L + 2\pi \int_0^S \lambda\alpha\gamma s\,(S-s)\,ds = \frac{3-2\lambda}{3}\pi\alpha\gamma S^3.
\end{aligned}
$$

Hence:

Proposition 3 *The relation between total commuting cost, the total differential land rent, and the total differential wage sum is:*

$$TDR = \frac{1-\lambda}{3-2\lambda} TDW = \frac{1}{2} TC.$$

For $\lambda = 0$, we obtain Proposition 1, for the case that the whole workforce works in the CBD.

Claim 4 *These expression are first order Taylor approximations of the relations that hold in more general models, where commuting cost vary non-linearly with distance and where the land intensity is negatively related to the land rent.*

3 Case 3: graphical exposition with variabel lot size in a 1-D city

We consider the case of an one-dimensional city, since this allows us to draw pictures. Suppose that lot size is variable. For simplicity, landintensity at the city edge α, travell cost per unit of distance γ, and the productivity in the city centre ω are normalized to unity. Hence:

$$S = \omega/\gamma = 1.$$

The slope of the rent function must be increasing in abolute value closer to the CBD. We take a convenient functional form:

$$R(s) = \exp(s - S) - 1 = \exp(s-1) - 1.$$

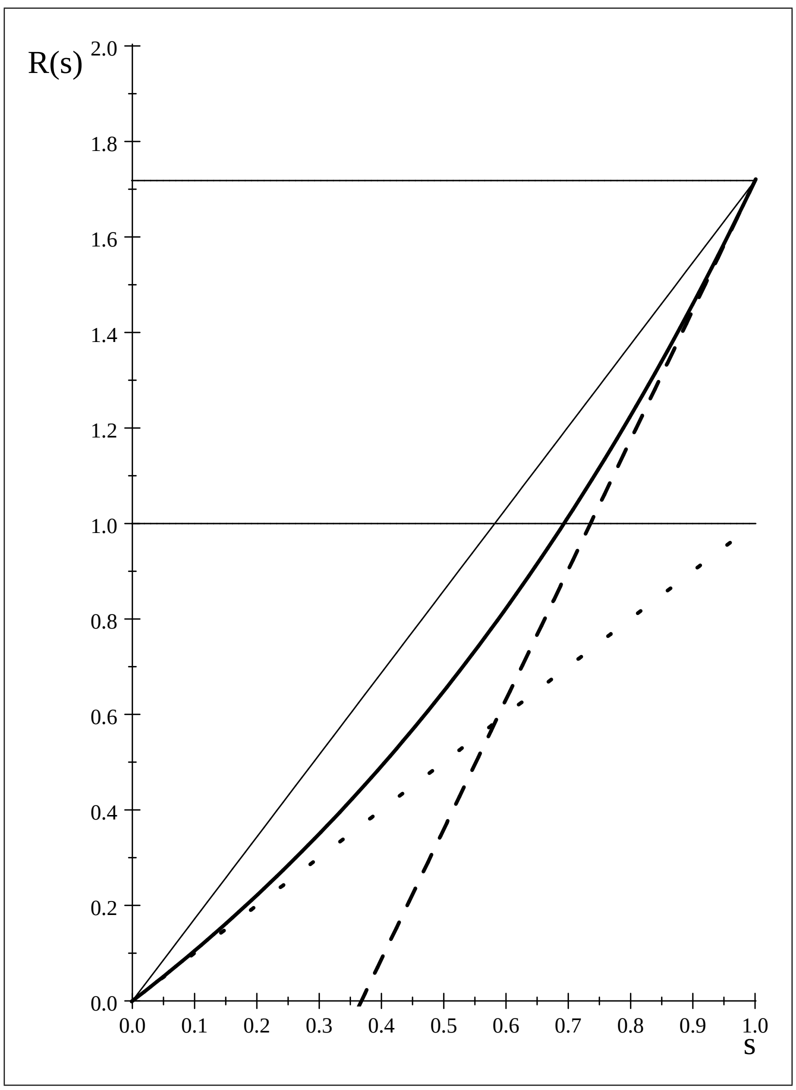
Our argument applies to any functional form with $R'(S) = 1, R''(s) > 0$. The utility equivalence result implies that $-R'(s)$ equals the population density at location s, since the land rent gradient is commuting cost advantage per unit of distance divided by the population density. Hence, the total polulation of the city is $2R(0)$. The benefit of the public good is unity. Hence, the total gross benefit of the city, population x benefit, is $2R(0)$. The total land rent satisfies:

$$TDR = 2\int_0^S R(s)\,ds.$$

Transport cost at location s is s. Hence, total transport cost satisfies:

$$TC = 2\int_0^S -sR'(s)\,ds = -\left[2sR(s)\right]_0^S + 2\int_0^S R(s)\,ds$$

Total transport cost is equal to the differential land rent. The figure sketches the situation.



Total gross benefit is the surface under the rectangle $x \in [0,1], y \in [0,e]$. Total land rent TDR is the twice area under the curve $R(s)$. Total transport cost is the same. Hence, the difference between total gross benefit on the one hand and total land rent plus total transport cost is twice the area between the diagonal and the curve $R(s)$, $R(S) - TDR$. Consider a person living a location s. It must be indifferent between living at the city edge S or at location s. Land rent per person at location s is $-R(s)/R'(s)$, while travell cost is s. Land rent at location S is zero, while travell cost is $S = 1$. The difference between both expressions, $1 + R(s)/R'(s) - s > 0$ is the compensation an individual at location s receives for the fact that she lives on a smaller plot of land. Integrating this compensation over the population reads:

$$\int_0^S -[1 + R(s)/R'(s) - s]\, R'(s)\, ds = \int_0^S -R'(s) - R(s) + sR'(s)\, ds = R(S) - 2Q.$$

Hence, the difference between the total gross benefit and the sum of travell cost and land rent is equal to the compensation that has to be paid to intramarginal inhabitants of the city for living a smaller plot of land. The net benefit of the public good is therefor equal to the total differential land rent.